# On the Brachistochrone Problem

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# Outline

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## Formulating the Brachistochrone Problem

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at $A$ and reaches $B$ in the shortest time.

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### Formulating the Brachistochrone Problem

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We will, of course, be presenting a solution based upon the Calculus of Variations in this presentation.

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2. Additionally, we suppose $y > y'$ for the length of the curve
3. and, as with many physical problems with analytical solutions, we choose to disregard friction. (At least initially)

# A start

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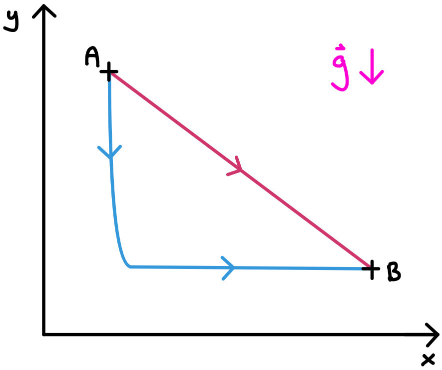
We should begin by formulating exactly what this curve may look like by equating the relevant forces, though we may happen upon two (or three) solutions by intuition. Given that we are attempting to minimise time, and we know

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

so of course we could try minimising either the distance travelled, or maximise the speed at which we travel.

# Min-maxing Speed and distance

In this visualisation, we have in red the shortest distance between $A$ and $B$ and then in blue we have some version of the ‘fastest velocity’, where $\vec{g}$ acts greatest at the start providing plenty of acceleration.



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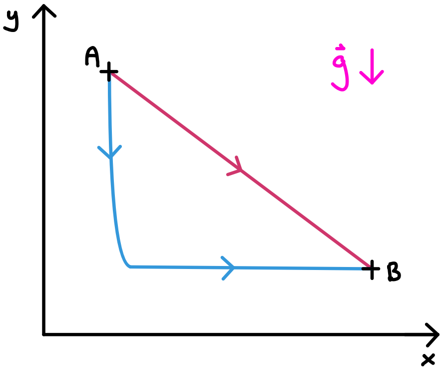


Unfortunately, neither of these happen to be optimal solutions to the problem, though we can now turn to the Calculus of Variations.

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and by our framing of the situation we have that $h = y(x)$ and so

$$v(s) = \sqrt{2g\, y(x)}$$

Then we can start by considering the time taken to travel along the curve $y = y(x)$ as a functional:

$$T[y] := \int_A^B \mathrm{d}t = \int_A^B \frac{\mathrm{d}s}{(\mathrm{d}s/\,\mathrm{d}t)} = \int_A^B \frac{\mathrm{d}s}{v(s)}$$

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Then given that $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ we arrange to obtain $\mathrm{d}s = \sqrt{1 + (y')^2}$ and use our $v(s) = \sqrt{2g\,y(x)}$

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$$\int_0^a \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g\,y(x)}}\,\mathrm{d}x = T[y]$$

and the attempt to minimise this functional is equivalent then to the Brachistochrone problem

# Outline

# Necessary Results

## The First Variation

We define a functional to be of a specific form, namely:
Let $J : \mathcal{C}_2[a, b] \to \mathbb{R}$ be a functional of the form

$$J[y] = \int_a^b f(x, y, y')\,\mathrm{d}x$$

where $f$ has continuous second order partial derivatives with respect to all its arguments.

# Necessary Results 2

## The Euler-Lagrange Equation

Let $J : \mathcal{C}_2[a, b] \to \mathbb{R}$ be a functional of the form

$$J[y] = \int_a^b f(x, y, y')\,\mathrm{d}x$$

Let $S = \{y \in \mathcal{C}_2[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\}$ where $(y_0, y_1) \in \mathbb{R}^2$
Then if $J[y]$ is 'stationary' at $y \in S$ we have:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \quad \forall x \in [a, b]$$

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# A Solution

The Brachistochrone functional

$$f(x, y, y') = \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g\, y(x)}}$$

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We observe that there is no explicit $x$ component to $f$ and now introduce a corollary of the Euler-Lagrange equation, namely:

## Beltraimi Identity

For functionals of the form

$$J[y] = \int_a^b f(y, y')\,\mathrm{d}x$$

the Euler-Lagrange equation reduces to the Beltrami identity:

$$f - y'\frac{\partial f}{\partial y'} = \text{Constant}$$

# A Solution

$$f(x, y, y') = \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g\, y(x)}}$$

So computing from here we have that

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{(1 + (y')^2)\, 2g\, y}}$$

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Hence, by the Beltrami identity we have:

$$f - y' \frac{\partial f}{\partial y'} = \mathrm{C}$$

$$\frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g\, y(x)}} - \frac{(y')^2}{\sqrt{(1 + (y')^2)\, 2g\, y}} = C$$

$$\implies C = \frac{1}{\sqrt{(1 + (y')^2)\, 2g\, y}}$$

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Squaring both sides and rearranging we get:

$$\left[ y + y \left( \frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 \right] = \frac{1}{2gC^2} = 2k$$

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Equivalently:

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for $k > 0$ a constant. Which provided with $y(0) = 0$ can be solved parametrically by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k(t - \sin(t)) \\ k(1 - \cos(t)) \end{pmatrix}$$

# Validating the solution

We now will check this solution by investigating its “second variation”

## The Second Variation

Let $J[y] = \int_{x_0}^{x_1} f(x, y, y')\,\mathrm{d}x$, where $y(x_0) = y_0, y(x_1) = y_1$ and $f$ is thrice continuously differentiable in all its arguments, then:

$$\delta^2 J[\eta, y] = \int_{x_0}^{x_1} \left\{ \eta^2 \left( \frac{\partial^2 f}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial^2 f}{\partial y \partial y'} \right) + (\eta')^2 \frac{\partial^2 f}{\partial y' \partial y'} \right\} \mathrm{d}x.$$

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## Minimum condition

Introducing $H = \{\eta \in \mathcal{C}_2[x_0, x_1] : \eta(x_0) = \eta(x_1) = 0\}$ we have that if $y$ is a local minimum then

$$\delta^2 J[\eta, y] \geq 0, \ \forall \eta \in H$$

# Validating the solution

First remarking that it should make no difference should we ignore the constant $\sqrt{2g}$ found in the integrand of the $T[y]$ functional and then:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{y(1+(y')^2)}}, \quad \frac{\partial f}{\partial y} = -\frac{f}{2y}$$

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Then we compute the second derivatives

$$\frac{\partial^2 f}{\partial y' \partial y'} = \frac{1}{\sqrt{y(1+(y')^2)^3}}, \quad \frac{\partial^2 f}{\partial y \partial y'} = -\frac{1}{2y}\frac{\partial f}{\partial y'}, \quad \frac{\partial^2 f}{\partial y \partial y} = \frac{3f}{4y^2}$$

# Validating the solution

Ultimately, we can rewrite

$$\delta^2 J[\eta, y] = \int_{x_0}^{x_1} \{\eta^2(\alpha) + (\eta')^2(\beta)\}\,\mathrm{d}x.$$

for $\alpha = \left(\frac{\partial^2 f}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial^2 f}{\partial y \partial y'}\right)$ and $\beta = \frac{\partial^2 f}{\partial y' \partial y'}$ and then by computation resolve that for

$$\alpha = \left.\frac{1}{2y^2\sqrt{y(1+(y')^2)}}\right|_{y=y_0} > 0, \quad \beta = \left.\frac{1}{\sqrt{y(1+(y')^2)^3}}\right|_{y=y_0} > 0.$$

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And so $\delta^2 T[\eta, y_0] \geq 0$ and hence by our previous remark, we have that the cycloid is at least a local minimiser of $T[y]$.

# With Friction

I will not be discussing the case for friction here but a solution relying on the Calculus of Variations can be found here:

N. Ashby, W. E. Brittin, W. F. Love, and W. Wyss. Brachistochrone with Coulomb friction.
*Amer. J. Phys.*, 43(10):902–906, 1975.

And that concludes the presentation. Thank you for your time