Euler Characteristics of Finite Categories

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What's an Euler Characteristic?

To start, we introduce the Euler Characteristic, χ as a 'topological invariant of finite simplicial complexes'.

Definition

A finite simplicial complex is a finite set X with a collection $\mathbb{A}$ of non-empty subsets of X such that

1. if $A \in \mathbb{A}$ and $B \subset A$ then $B \in \mathbb{A}$ and,
2. for each $x \in X$ the singleton $\{x\} \in \mathbb{A}$.

So that we can think of X as a set of vertices with subsets of size $k+1$ in $\mathbb{A}$ being called k-simplices.

Example

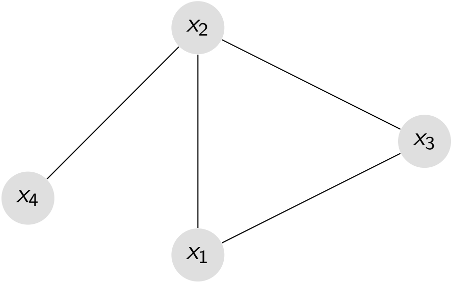

Figure: A set X of 4 vertices.

With collection:

$$\mathbb{A} = \begin{cases} \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \\ \{x_4, x_2\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \\ \{x_1, x_2, x_3\} \end{cases}$$

Euler Characteristic of a finite simplex

We can define the Euler characteristic on a finite simplicial complex with the following formula

$$\chi(X) = \sum_{A \in \mathbb{A}} (-1)^{|A|-1}$$

So that in our example we have that $\chi(X) = 4 - 4 + 1 = 1$.

Rota's Contribution

In 1964 Rota extended the Euler characteristic to finite posets. He noted that we can define the collection $\mathbb{A}$ from finite simplicial complexes as a sub-poset of $(2^X, c)$.

Figure: Rota sometime later in 1970

Euler Characteristic of a Poset

Consider a sequence of points, $p_0 < p_1 < \ldots < p_n$ of length n in P. We call such a sequence a chain. Then a finite poset, P, has Euler characteristic:

$$\chi(P) = \sum_{n \in \mathbb{N}} (-1)^{\#\text{chains of length } n}$$

χ from μ

Rota also showed us that Euler characteristics could be derived from "Möbius Inversion" in posets.

Defining a new poset $\bar{P}$ of P with an additional minimal element 0 and maximal element 1. Then

$$\chi(P) = \mu_{\bar{P}}(0,1) + 1.$$

Which connected existing theory on incidence algebras and the Phillip Hall theorem from combinatorics relating $\mu(0,1)$ of an incidence algebra to the number of chains.

Leinster and onward

So in the late naughties, Tom Leinster produced his paper relating Euler characteristics to finite categories. Which we ought to define here as they will be used for the remainder of the presentation.

Definition

A finite category $\mathbb{A}$ is

1. A finite collection $\mathrm{Ob}(\mathbb{A})$ of objects,
2. A finite collection of arrows (or morphisms) $\mathbb{A}(a, b)$ for each $a, b \in \mathrm{Ob}(\mathbb{A})$ that satisfy a composition law, i.e. for each $a, b, c \in \mathrm{Ob}(\mathbb{A})$ a map

$$\mathbb{A}(a, b) \times \mathbb{A}(b, c) \to \mathbb{A}(a, c)$$

3. An identity arrow is also required, $1_a \in \mathbb{A}(a, a)$ such that $1_a \circ \theta = \theta$ for all $\theta \in \mathbb{A}(b, a)$ and similarly $\theta \circ 1_a = \theta \; \forall \theta \in \mathbb{A}(a, b)$.

The $R(\mathbb{A})$ $\mathbb{Q}$-Algebra

Definition

Let $R(\mathbb{A})$ be the $\mathbb{Q}$-algebra of functions $\mathrm{Ob}(\mathbb{A}) \times \mathrm{Ob}(\mathbb{A}) \to \mathbb{Q}$ with point-wise addition and convolution as product. That is for functions $\theta, \phi \in R(\mathbb{A})$ and objects $a, b, c \in \mathrm{Ob}(\mathbb{A})$ we have the following properties

- $(\theta + \phi)(a, b) = \theta(a, b) + \phi(a, b)$ as *addition*,
- $(k \cdot \theta)(a, b) = k \cdot \theta(a, b)$ for $k \in \mathbb{Q}$, as *scalar multiplication*,
- Then finally, we have *convolution* defined by:

$$(\theta \cdot \phi)(a, c) = \sum_{b \in \mathrm{Ob}(\mathbb{A})} \theta(a, b)\phi(b, c).$$

Special functions

The multiplicative identity element of the incidence algebra is denoted by the **Kronecker Delta** $\delta(a, b) \in R(\mathbb{A})$ and is evaluated as follows:

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

We also have the **zeta function**:

$$\zeta(a, b) = |\mathbb{A}(a, b)|.$$

If ζ is invertible in $R(\mathbb{A})$, then the category has **Möbius inversion**:

$$\mu_{\mathbb{A}} = \mu = \zeta^{-1}.$$

$$\sum_{b \in \mathrm{Ob}(\mathbb{A})} \mu(a, b)\zeta(b, c) = \delta(a, c) = \sum_{b \in \mathrm{Ob}(\mathbb{A})} \zeta(a, b)\mu(b, c) \quad \forall a, c \in \mathbb{A}.$$

Existence of μ

Unfortunately, we don't always get a μ for a given category. So some conditions must be put in place. Namely that we need a type of category called a 'skeletal' category:

Definition (Skeletal Category)

A finite category $\mathbb{A}$ is skeletal if and only if all isomorphisms are identities.

Note: $a = b \implies a \cong b$ always, whilst $\mathbb{A}$ skeletal such that $a \cong b \implies a = b$.

A formula for μ

Though we won't prove or fully explain it today, we can take this skeletal property in conjunction with a handful of other properties to produce a formula for computing the Möbius inversion of a finite category $\mathbb{A}$:

Theorem

Let $\mathbb{A}$ be a finite skeletal category in which the only idempotents are identities. Then $\mathbb{A}$ has Möbius inversion given by

$$\mu(a,b) = \sum_{\substack{n \geq 0 \\ \text{paths } a \to b}} \frac{(-1)^n}{|Aut(a_0)| \cdots |Aut(a_n)|}$$

for $a = a_0$ and $b = b_n$ with $Aut(a)$ being the automorphism group of $a \in \mathbb{A}$ and where the sum runs over all $n \geq 0$ and paths for which $a_0, \ldots, a_n$ are pairwise distinct.

Leinster's Weightings

One of Leinster's compelling contributions was the development of the 'weighting' and 'coweighting' on a finite category.

Definition

Let $\mathbb{A}$ be a finite category. A **weighting** on $\mathbb{A}$ is a function $k^\bullet : \mathrm{Ob}(\mathbb{A}) \to \mathbb{Q} : \forall a \in \mathbb{A}$,

$$\sum_b \zeta(a,b) k^b = 1$$

where $\zeta(a,b) = |\mathbb{A}(a,b)|$ and $k^\bullet$ denotes the 'weight' of $\bullet \in \mathrm{Ob}(\mathbb{A})$.

Relating Weighting and Inversion

Lemma

$\mathbb{A}$ has Möbius inversion $\iff$ $\mathbb{A}$ has a unique weighting $\iff$ $\mathbb{A}$ has a unique coweighting; they are given by:

$$k^a = \sum_b \mu(a, b), \quad k_b = \sum_a \mu(a, b)$$

Weightings and Coweightings

Lemma

Let $\mathbb{A}$ be a finite category with weightings and coweightings $k^\bullet$ and $k_\bullet$ respectively. Then $\sum_a k^a = \sum_a k_a$.

Proof.

$$\sum_b k^b = \sum_b \left(\sum_a k_a \zeta(a, b) \right) k^b = \sum_a k_a \left(\sum_b \zeta(a, b) k^b \right) = \sum_a k_a \quad \square$$

Computing Weightings

By hand! Boo!

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Euler Characteristic χ

Definition

A finite category $\mathbb{A}$ has Euler characteristic if it admits both a weighting and coweighting. Its Euler characteristic is then

$$\chi(\mathbb{A}) = \sum_a k^a = \sum_a k_a \in \mathbb{Q}$$

for any weighting $k^\bullet$ and coweighting $k_\bullet$.

Computing Euler Characteristic χ

By hand! Boo!

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The End!