MATH552:
Euler Characteristics of Finite Categories

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Contents

1 Introduction

We may begin by introducing Euler characteristics as a topological invariant on finite simplicial complexes, depicting k-simplices as being those subsets of size $k+1$ of a collection $(\mathbb{A})$ of non-empty subsets of some finite set X. Then we define the Euler characteristic of X as follows: $\chi(X) = \sum_{A \in \mathbb{A}} (-1)^{|\mathbb{A}|-1}$, the alternating sum of the number of k simplices. This simple formula captures the 'vertices minus edges plus faces' formula that first exposes many of us to χ on simple 2D and 3D shapes.

Though, this dimensionless property extends to many other formulations and reaches rather widely into different areas of mathematics. For example, one may encounter Euler characteristics in theorems such as the Gauss-Bonnet theorem, linking χ to differential geometry and smooth manifolds or perhaps from the topological sense learning of genus' and later applying this in more unusual or niche settings such as complex analysis via the Riemann-Roch theorem or perhaps more directly the Riemann-Hurwitz formula. In addition, as Leinster points out in [Lei08], it is also true that Euler Characteristics can be tied to combinatorics via Möbius inversion of posets as was presented by Gian-Carlo Rota and contemporaries. One can even find links to crystallography via the late John Conway's influence; making use of orbifolds and their Euler characteristics to discuss the relevant 17 wallpaper groups.

Given the pervasiveness of this property, it is no surprise then that we should look to learn of more novel contexts in which χ exists. This paper is modelled after Leinster's ([Lei08]) which was similarly motivated and looks to extend Euler characteristics to 'finite categories'.

Leinster's paper was, however, targeted at an audience already familiar with categories. We instead take the route of walking through and introducing much of the prerequisite Category Theory as it pertains to the upcoming theory as presented by Leinster. In addition, we provide further exposition to many of the proofs along the way to better guide the reader through the logic and arguments present in the initial paper which was rather brief at times.

In this paper, we shall walk through definitions of finite categories, factorisation systems, numerous types of functors, equivalences and adjoints, to name a few, all in service of proving facts about Möbius inversion as it applies to categories and to build up the required theory of 'weightings' and 'coweightings' required to construct the Euler characteristic of a finite category. Along the way we shall present expository examples and discuss some of the motivations behind the theory being presented. In the final chapter we present colimits and discuss how the theory presented can be applied right away to inform us of the cardinality of colimits.

Though not necessarily referenced directly, both [Lei14] and [Sim11] as well as [Lan78] were tremendously useful in the process of developing an understand of the underlying category theory.

In Appendix A one will find MAPLE procedures that automate the necessary calculations of 'weightings' as well as that of Euler characteristics of finite categories. As there does not seem to be any well supported method of inputting syntax-highlighted MAPLE code into LaTeX, I have produced a language style file to work with the `listings` package to closely match that of the MAPLE's code-edit environment.

2 Categories and Möbius Inversions

2.1 Categories

We begin by introducing finite categories.

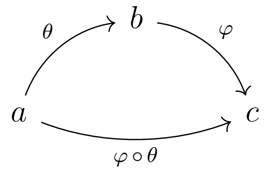
Definition 2.1 (Finite Category)**.** A finite category $\mathbb{A}$ is

1. A finite collection $\mathrm{Ob}(\mathbb{A})$ of objects,

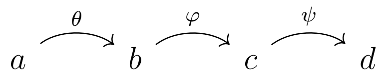
2. A finite collection of arrows (or morphisms) $\mathbb{A}(a, b)$ for each $a, b \in \mathrm{Ob}(\mathbb{A})$ that satisfy a composition law, i.e. for each $a, b, c \in \mathrm{Ob}(\mathbb{A})$ a map

$$\mathbb{A}(a, b) \times \mathbb{A}(b, c) \to \mathbb{A}(a, c)$$

   which we can visualise as



   which is associative, namely, given $\theta : a \to b$, $\varphi : b \to c$ and $\psi : c \to d$ we have:



   such that $\psi \circ (\varphi \circ \theta) = (\psi \circ \varphi) \circ \theta$,

3. An identity arrow is also required, $1_a \in \mathbb{A}(a, a)$ such that $1_a \circ \theta = \theta$ for all $\theta \in \mathbb{A}(b, a)$ and similarly $\theta \circ 1_a = \theta \ \forall \theta \in \mathbb{A}(a, b)$.

Before moving into some examples of categories we quickly discuss **endomorphisms**.

Remark 2.2. An endomorphism of $a \in \mathrm{Ob}(\mathbb{A})$ is an arrow in $\mathbb{A}(a, a)$. If an endomorphism θ is invertible, that is, if

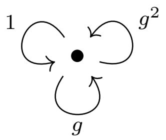
$$\exists \theta^{-1} \in \mathbb{A}(a, a) : \begin{cases} \theta \circ \theta^{-1} = 1_a \\ \theta^{-1} \circ \theta = 1_a. \end{cases}$$

then we call θ an **automorphism**. We denote the subset of automorphisms of a by $\mathrm{Aut}(a) \subset \mathbb{A}(a, a)$.

We can now begin to look at some simple examples.

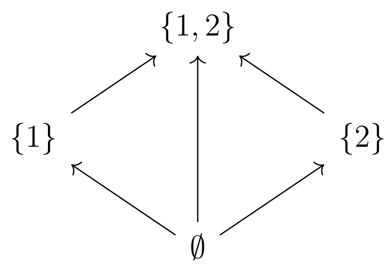
Example(s) 2.3. Here are two examples of finite categories.

1. Consider a finite group G, then we may consider it as a category, $\mathbb{G}$, consisting of a single object $\bullet \in \mathrm{Ob}(\mathbb{G})$. Given that we only have one object we can observe that any arrow must take this object to itself and is thus an endomorphism. Every endomorphism of $\bullet$ is an automorphism, i.e. $\mathbb{G}(\bullet, \bullet) = \mathrm{Aut}(\bullet) = G$. We note then that any composition of arrows is defined. Taking G to be C_3, the cyclic group of order 3, consisting of elements $\{1, g, g^2\}$. Then we may portray the corresponding category $\mathbb{G}_{C_3}$ in the following diagram:

   with arrows representing maps from $\bullet \to \bullet$. Alternatively, as is portrayed, the different elements of the group.

2. Consider a category $\mathbb{A}$ consisting of the following objects for some fixed $n \in \mathbb{N}$,
   $$\mathrm{Ob}(\mathbb{A}) = \{\text{subsets of } \{1, \dots, n\}\}.$$
   The arrows are given by
   $$\mathbb{A}(a, b) = \begin{cases} \emptyset & \text{if } a \not\subset b \\ \text{has one element} & \text{if } a \subset b \end{cases}$$
   which also uniquely determines the composition law. Suppose we fix $n = 2$ to be our example. Then the category has objects $\mathrm{Ob}(\mathbb{A}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and the following diagram depicts all non-identity arrows,

   From this diagram we can see that there is a unique composition law.

2.2 Möbius inversion

To begin, we introduce the finite incidence algebra $R(\mathbb{A})$ of a finite category $\mathbb{A}$. When the category $\mathbb{A}$ is evident, we simply write R.

Definition 2.4 (The $R(\mathbb{A})$ $\mathbb{Q}$-Algebra)**.** Let $R(\mathbb{A})$ be the $\mathbb{Q}$-algebra of functions $\mathrm{Ob}(\mathbb{A}) \times \mathrm{Ob}(\mathbb{A}) \to \mathbb{Q}$ with point-wise addition and convolution as product. That is for functions $\theta, \phi \in R(\mathbb{A})$ and objects $a, b, c \in \mathrm{Ob}(\mathbb{A})$ we have the following properties

- $(\theta + \phi)(a, b) = \theta(a, b) + \phi(a, b)$ as *addition*,
- $(k \cdot \theta)(a, b) = k \cdot \theta(a, b)$ for $k \in \mathbb{Q}$, as *scalar multiplication*,
- Then finally, we have *convolution* defined by:

$$(\theta \cdot \phi)(a, c) = \sum_{b \,\in\, \mathrm{Ob}(\mathbb{A})} \theta(a, b)\phi(b, c).$$

The multiplicative identity element of the incidence algebra is denoted by $\delta(a, b) \in R(\mathbb{A})$ and is evaluated as follows:

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

So taking a convolution involving $\delta(a, b)$ we may eliminate all sums over $b : b \neq c$ (as these would involve multiplying by a scalar zero), giving $(\theta \cdot \delta)(a, c) = \sum_{b \,\in\, \mathrm{Ob}(\mathbb{A})} \theta(a, b)\delta(b, c) = \theta(a, c)$. Similarly $\delta \circ \theta = \theta$.

Remark 2.5. If we take some arbitrary ordering of the finite number of objects of $\mathbb{A}$, the incidence algebra is isomorphic to the algebra of matrices with elements in $\mathbb{Q}$ of size $|\mathrm{Ob}(\mathbb{A})| \times |\mathrm{Ob}(\mathbb{A})|$ with matrix multiplication as products, which is to say explicitly: $R(\mathbb{A}) \cong M_{|\mathrm{Ob}(\mathbb{A})| \times |\mathrm{Ob}(\mathbb{A})|}(\mathbb{Q})$.

Remark 2.6 (Functions in the $R(\mathbb{A})$ $\mathbb{Q}$-algebra). Within $R(\mathbb{A})$ we also have the **zeta function** which is defined to be $\zeta(a, b) = |\mathbb{A}(a, b)|$: denoting the number of arrows between objects a and b. If ζ is invertible in $R(\mathbb{A})$, then we say that the category has **Möbius inversion** and denote the inverse as $\mu_{\mathbb{A}} = \mu = \zeta^{-1}$.

Taking note that the Möbius inversion inverts the zeta function, their convolution is the Kronecker-δ. Explicitly:

$$\sum_{b \,\in\, \mathrm{Ob}(\mathbb{A})} \mu(a, b)\zeta(b, c) = \delta(a, c) = \sum_{b \,\in\, \mathrm{Ob}(\mathbb{A})} \zeta(a, b)\mu(b, c) \quad \forall a, c \in \mathbb{A}.$$

In Example 2.3-(1) we introduced a finite category $\mathbb{G}$, with one object so $R(\mathbb{G}) = \mathbb{Q}$. The zeta function is $\zeta(\bullet, \bullet) = |G|$ (the size of the group) with $\delta(\bullet, \bullet) = 1_G$ and so we may compute $\mu_{\mathbb{G}} = \zeta^{-1} = \frac{1}{|G|}$.

We now introduce skeletal categories, first by making note of what is meant by an isomorphism between objects.

Definition 2.7 (Isomorphic Objects)**.** We say $a, b \in \mathrm{Ob}(\mathbb{A})$ are isomorphic if there are arrows of the form

$$a \underset{\varphi \in \mathbb{A}(b,a)}{\overset{\theta \in \mathbb{A}(a,b)}{\rightleftarrows}} b$$

such that $\theta \circ \varphi = 1_b$ and $\varphi \circ \theta = 1_a$. If a and b are isomorphic we write $a \cong b$, with $a \cong a, \forall a \in \mathrm{Ob}(\mathbb{A})$ via the identity.

Again, note that in Example 2.3-(1) all arrows are isomorphisms, because each element in a group G has an inverse. So $\bullet \cong \bullet$ via every morphism in $\mathbb{G}(\bullet, \bullet)$.

We can now define skeletal categories which play a vital role.

Definition 2.8 (Skeletal Category)**.** A finite category $\mathbb{A}$ is skeletal if and only if all isomorphisms are identities. Note: $a = b \implies a \cong b$ always, whilst $\mathbb{A}$ skeletal such that $a \cong b \implies a = b$.

However as we have shown for $\mathbb{G}$ above, we can have non skeletal categories such that $a \cong b \iff a = b$ holds.

Our motivation for introducing skeletal categories is so that we may combine Remark 2.5 and Definition 2.8. As a skeletal category has no redundant isomorphisms we see that there are no duplicate rows in the matrix of ζ.

2.3 Finding Möbius Inversions

It is not such a simple task to find the Möbius inversions of some finite categories without first developing some prerequisite tools. We were in good luck that it happened to be simple for the category defined from a finite group, $\mathbb{G}$, but we are not always so fortunate, nor should we be so bold as to assume that such an inversion can be found at all! Hence the remainder of this section will be used to introduce and demonstrate a handful of techniques which will empower us to find these inversions.

Definition 2.9 (n-paths, [Lei08])**.** Let $n \geq 0$ and let $\mathbb{A}$ be our finite category, with $a, b, \in \mathrm{Ob}(\mathbb{A})$. An n-path from $a \to b$ is the diagram in $\mathbb{A}$

$$a = a_0 \xrightarrow{\theta_1} a_1 \xrightarrow{\theta_2} \dots \xrightarrow{\theta_n} a_n = b$$

This is a **circuit** if $a = b$ and is **non-degenerate** if no $\theta_i \in \mathrm{Ob}(\mathbb{A})$ is an identity.

Then before we introduce the following lemma we define what is meant for an arrow to be idempotent.

Definition 2.10. An arrow $\theta \in \mathbb{A}(a, a)$ is idempotent if $\theta \circ \theta = \theta$.

Lemma 2.11 ([Lei08])**.** *The following conditions on a finite category, $\mathbb{A}$, are equivalent:*

a. Every idempotent is an identity.

b. Every endomorphism in $\mathbb{A}$ is an automorphism.

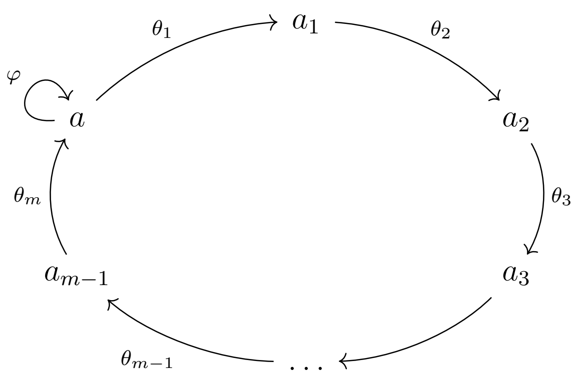
c. Every circuit in $\mathbb{A}$ consists entirely of isomorphisms.

Proof: (a. $\implies$ b.). Suppose $\varphi \in \mathbb{A}(a, a)$ is an endomorphism. By the Pigeonhole principle, there exists some $m \in \mathbb{N}$ and infinitely many $n \in \mathbb{N}$ such that $\varphi^n = \varphi^m$. Let us choose $n > 2m$ with $\varphi^n = \varphi^m$. Then

$$\begin{aligned} \varphi^{(n-m)m} &= (\varphi^m)^{n-m} \\ &= (\varphi^m)^m \cdot (\varphi^m)^{n-2m} \\ &= \varphi^{2mn-2m^2} \\ &= \varphi^{2m(n-m)} \\ &= \left(\varphi^{m(n-m)}\right)^2 \end{aligned}$$

Hence $\varphi^{(n-m)m}$ is an idempotent. Hence $\varphi^{(n-m)m} = \mathrm{id}_a$. Therefore φ is an isomorphism (with inverse $\varphi^{(n-m)m-1}$).

Proof: (b. $\implies$ c.). Assume every endomorphism is an automorphism. If we have a circuit (equivalently an n-path whose composite is an endomorphism) such that $\theta_m \circ \theta_{m-1} \circ \ldots \theta_1 \in \mathbb{A}(a,a)$,



Then by assumption this is an automorphism i.e. $\exists \varphi \in \mathbb{A}(a,a)$ with

$$(\varphi \circ \theta_m \circ \theta_{m-1} \circ \ldots) \circ \theta_1 = 1_a.$$

So we see that θ_1 has a left-inverse. A similar argument considering $\theta_1 \circ \theta_m \circ \cdots \circ \theta_2$ shows that θ_1 also has a right-inverse. Hence we can see that each θ_i has both left and right inverse.

Proof: (c. $\implies$ a.). Suppose $\varphi \in \mathbb{A}(a,a)$ is an idempotent, i.e. $\varphi \circ \varphi = \varphi$ and consider a circuit with one arrow.

Then by assumption this circuit consists entirely of isomorphisms. Hence by invertibility φ is self inverse and $\varphi = \varphi \circ \varphi \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = 1_a$. So each idempotent is an identity. □

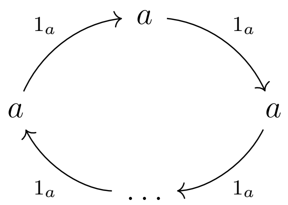
We now introduce the following Theorem 2.12 that gives us one method of identifying Möbius inversions for skeletal categories.

Theorem 2.12. *Let $\mathbb{A}$ be a finite skeletal category (Definition 2.8) in which the only idempotents are identities (Lemma 2.11 (a.)). Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum_{\substack{n \geq 0 \\ \text{paths } a \to b}} \frac{(-1)^n}{|Aut(a_0)| \cdots |Aut(a_n)|}$$

for $a = a_0$ and $b = b_n$ with $Aut(a)$ being the automorphism group of $a \in \mathbb{A}$ and where the sum runs over all $n \geq 0$ and paths (Definition 2.9) for which $a_0, \ldots, a_n$ are pairwise distinct.

Proof. Since $\mathbb{A}$ is skeletal, and all idempotents are identities, we conclude that all circuits must be of the form:



As there are no extraneous isomorphisms ($\mathbb{A}$ is skeletal) and we have that $\theta \circ \theta = \theta$ for $\theta = \mathbb{A}(a, a)$ (from idempotency) it must be the case that the only arrows formed along a path create a circuit with all $\theta_1 = \cdots = \theta_n = 1_a$. Thus there are no non-degenerate circuits. We now recall that we are in the $R(\mathbb{A})$ $\mathbb{Q}$-algebra (as in Definition 2.4) and introduce the following result to assist in the proof.

Lemma 2.13. *Let $\mathbb{A}$ be a finite skeletal category in which the only idempotents are identities. Then:*

$$\sum_{\substack{b:b\neq c,\\ g\in\mathbb{A}(b,c)}} \frac{\mu(a,b)}{|Aut(c)|} = \delta(a,c) - \mu(a,c)$$

for $a, c \in \mathbb{A}$.

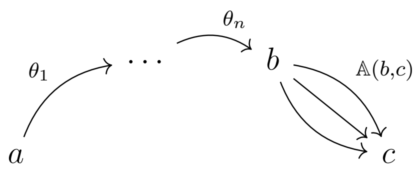
Proof of Lemma 2.13. We first start by presenting a visualisation of the possible paths from $a \to c$

1. We have direct paths

$$a \longrightarrow c$$

   of which there are $|\mathbb{A}(a, c)|$, or

2. We have n-paths (possibly of length 0; i.e $a = b$ in the following diagram) which are of the form:



   with there being $|\mathbb{A}(b, c)|$-many paths from b to c and n-paths from a to b.

With this in mind, our summation $\sum_{b:b\neq c, g\in\mathbb{A}(b,c)} \frac{\mu(a,b)}{|\mathrm{Aut}(c)|}$ is working over all such n-paths of length $n \geq 0$ and for the second kind of path as described above. Hence, by use of the fact that there are no non-degenerate circuits, we note that our summand has $\mu(a,b) = 0$ for all pairwise distinct $a_i \neq c$ and $\mu(a,b) = (-1)^i$ hence the sum is equal to $\frac{1}{|\mathrm{Aut}(a)|}$ when $a = c$. Which is to say that it is exactly equal to what we define as the Kronecker-delta for a and c minus the inversion for when they are equal, hence $\delta(a,c) - \mu(a,c)$. $\square$

Now we may return to our original proof and write out the convolution of μ and ζ as follows:

$$
\begin{aligned}
(\mu \cdot \zeta)(a,c) &= \sum_{b\in\mathbb{A}} \mu(a,b)\zeta(b,c) && (1)\\
&= \mu(a,c)\zeta(c,c) + \sum_{b\neq c} \mu(a,b)\zeta(b,c) && (2)\\
&= \mu(a,c)|\mathrm{Aut}(c)| + |\mathrm{Aut}(c)| \cdot \sum_{\substack{b:b\neq c\\ g\in\mathbb{A}(b,c)}} \frac{\mu(a,b)}{|\mathrm{Aut}(c)|} && (3)\\
&= |\mathrm{Aut}(c)| \left\{ \mu(a,c) + \sum_{b:b\neq c,\, g\in\mathbb{A}(b,c)} \frac{\mu(a,b)}{|\mathrm{Aut}(c)|} \right\} && (4)\\
&= |\mathrm{Aut}(c)| \left\{ \frac{\delta(a,c)}{|\mathrm{Aut}(c)|} \right\} && (5)\\
&= \delta(a,c) && (6)
\end{aligned}
$$

We begin in line (2) by pulling out the $b = c$ term from our summation. Then we note in line (3) that any endomorphism $\zeta(c,c)$ of c is going to be an automorphism by Lemma 2.11. So we may write $\zeta(c,c) = |\mathrm{Aut}(c)|$ then include it in and around our summand. In line (4) we factor out $|\mathrm{Aut}(c)|$ and note that as in Lemma 2.13 above we may collapse the summation into its form $\delta(a,c) - \mu(a,c)$ as in (5). Then as expected, $(\mu \cdot \zeta)(a,c) = \delta(a,c)$.

As a result, by acting over each distinct path $a_0 \neq \ldots \neq a_n \neq c$ we can rewrite the inversion part of the summand in the following alternative form:

$$\sum_{b:b\neq c, g\in\mathbb{A}(b,c)} \frac{\mu(a,b)}{|\mathrm{Aut}(c)|} = \sum \frac{(-1)^n}{|\mathrm{Aut}(a_0)| \ldots |\mathrm{Aut}(a_n)||\mathrm{Aut}(c)|}$$

This concludes what we wish to show. $\square$

Expanding on Theorem 2.12, we make use of the other equivalences in Lemma 2.11 to present the summation in another way.

Corollary 2.14 ([Lei08, Corollary 1.5])**.** *Let $\mathbb{A}$ be a finite skeletal category in which the only endomorphisms are identities (Lemma 2.11 (b.)). Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum_{n\geq 0} (-1)^n |\{\textit{non-degenerate } n\textit{-paths from } a \textit{ to } b\}| \in \mathbb{Z}$$

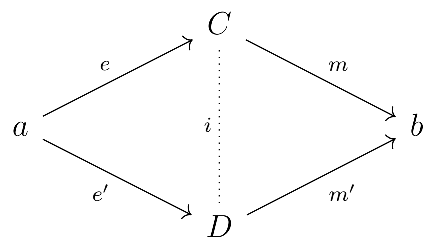
Proof. Non-degenerate paths are those for which no θ_i makes is an identity arrow within an n-path (Definition 2.9). By the equivalences present in Lemma 2.11 (b. and c.) we know that if every endomorphism is an automorphism (and also an identity) that any circuit contains distinct a_i with isomorphisms θ_i. □

Definition 2.15 (Epi-mono factorisation [nca])**.** Given a category $\mathbb{A}$ we take a "factorisation system, (E, M)" to be a pair of classes of maps, E and M such that every arrow $\theta \in \mathbb{A}(a,b)$ factors $\theta = r \circ l$ with the arrows $l \in E$ followed by $r \in M$ which must satisfy the following conditions:

1. The factorisation is unique up to isomorphism.
2. E and M contain all isomorphisms and are closed under composition.

If we take our left class to be the class of epimorphisms, $\mathcal{E}$, (right-cancellative arrows) and the right class that of monomorphisms, $\mathcal{M}$, (left-cancellative arrows), we obtain the **Epi-mono factorisation** system $(\mathcal{E}, \mathcal{M})$ for $\mathbb{A}$ with $\mathbb{A}(a,b) \ni \theta = me$ for $e \in \mathcal{E}$ and $m \in \mathcal{M}$. However, this may not always exist.

By the conditions of any factorisation system, we know that as the factorisation is unique up-to isomorphism that any other pair, i.e. $e' \in \mathcal{E}$ and $m' \in \mathcal{M}$, has unique isomorphism $i : C \to D$ with $\theta = me : a \to C \to b$ and $\theta = m'e' : a \to D \to b$ so that the following diagram commutes:



It may be useful later that as [Sim11, p.50] states, for 'appropriately nice categories' (often those for which the following notions are well understood), an injective map is a monomorphism while a surjective map is an epimorphism.

Theorem 2.16. *Let $\mathbb{A}$ be a finite skeletal category with an epi-mono factorisation system (Definition 2.15). Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum \frac{(-1)^n}{|Aut(a_0)| \cdots |Aut(a_n)|}$$

where the sum is over all $n \geq r \geq 0$ and paths (Definition 2.9) such that $a_0, \ldots, a_r$ are distinct, $a_r, \ldots, a_n$ are distinct with corresponding $f_1, \ldots, f_r \in \mathcal{M}$ and $f_{r+1}, \ldots, f_n \in \mathcal{E}$.

Proof. See [Lei08]. □

This gives us an alternate way of computing the Möbius inversion of categories so long as they hold an epi-mono factorisation (which is fairly common and often well known).

2.4 Application of the Möbius function

We now present definitions for more key concepts in category theory, namely that of functors. In particular, we introduce the notion of representable functors and adapt notation from [Lei14] to that which is more familiar to this paper. Together with the Möbius function we are able to determine the coefficients that define a familially representable functor in Proposition 2.19.

Definition 2.17 (Functors)**.** Let $\mathbb{A}$ and $\mathbb{B}$ be categories. A functor $X : \mathbb{A} \to \mathbb{B}$ consists of

1. A function
$$\mathrm{Ob}(\mathbb{A}) \to \mathrm{Ob}(\mathbb{B})$$
which is written as $A \mapsto X(A)$;
2. For each $A, A' \in \mathbb{A}$, a function
$$\mathbb{A}(A, A') \to \mathbb{B}(X(A), X(A')),$$
which is written as $f \mapsto X(f)$, satisfying the following axioms:
   (a) $X(f' \circ f) = X(f') \circ X(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathbb{A}$;
   (b) $X(1_A) = 1_{X(A)}$ whenever $A \in \mathbb{A}$.

For example if we take a functor $\mathbb{A} \xrightarrow{X} \mathbf{Set}$ this maps $a \in \mathrm{Ob}(\mathbb{A}) \to Xa$ (a set), and $\alpha \in \mathbb{A}(a,b) \to X\alpha : Xa \to Xb$ such that $X(\mathrm{id}_a) = \mathrm{id}(X_a)$, and $X(\beta \circ \alpha) = X\beta \cdot X\alpha$.

Definition 2.18 (Representable Functor)**.** Let $\mathbb{A}$ be a finite category with $a \in \mathbb{A}$. We introduce a functor $\mathbb{A}(a, -)$ from $\mathbb{A}$ to the category **Set** (the category whose objects are sets with arrows which are functions between sets):

$$\mathbb{A}(a, -) : \mathbb{A} \to \mathbf{Set}.$$

For a given object, $b \in \mathrm{Ob}(\mathbb{A})$, we 'fill in the blank' (denoted with '$-$') as $\mathbb{A}(a, b)$. Then, for a map $\theta \to \theta'$ we define

$$\mathbb{A}(a, \theta) : \mathbb{A}(a, b) \to \mathbb{A}(a, b')$$

by $- \mapsto \theta \circ -$, $\forall - : a \to b$. Then we say a functor is **representable** if $X \cong \mathbb{A}(A, -)$ for some $A \in \mathbb{A}$.

Building further, we call X **familially representable** if

$$X \cong \sum_{a \in \mathrm{Ob}()A} r(a)\mathbb{A}(a, -)$$

for some $r(a) \in \mathbb{N}$ where $\sum_a r(a)\mathbb{A}(a, -)$ denotes the disjoint union of $r(a)$ copies of the sets $\mathbb{A}(a, -)$ for $a \in \mathrm{Ob}(\mathbb{A})$.

Proposition 2.19. *Let $\mathbb{A}$ be a finite category with Möbius inversion and let $X : \mathbb{A} \longrightarrow \mathbf{Set}$ be a familially representable functor satisfying*

$$X \cong \sum_a r(a)\mathbb{A}(a, -)$$

for some natural numbers $r(a)$, $(a \in \mathbb{A})$. Then

$$r(a) = \sum_b |Xb|\mu(b, a) \quad \forall a \in \mathbb{A}.$$

Proof. By definition of convolution in the $R(\mathbb{A})$ $\mathbb{Q}$-algebra we remind ourselves of the following: $(\zeta\mu)(c, a) = \sum_{b \in \mathbb{A}} \zeta(c, b)\mu(b, a)$. Replacing the $\zeta(c, b)$ with $|\mathbb{A}(c, b)|$ we note $\sum_{b \in \mathbb{A}} |\mathbb{A}(c, b)|\mu(b, a) = \delta(c, a)$. Which is defined to be 1 if $c = a$ and 0 otherwise.

As above, we define a familially representable functor Xb to be of the form: $Xb = \sum_c r(c)\mathbb{A}(c,b)$. By explicit computation, we then show:

$$\begin{aligned}
\sum_b |Xb|\mu(b,a) &= \sum_b \left|\sum_c r(c)\mathbb{A}(c,b)\right| \mu(b,a) \\
&= \sum_b \left(\sum_c r(c)|\mathbb{A}(c,b)|\right) \mu(b,a) \\
&= \sum_c r(c) \sum_b |\mathbb{A}(c,b)|\mu(b,a) \\
&= \sum_c r(c)\delta(c,a) \\
&= r(a),
\end{aligned}$$

as desired. □

3 Weightings and Euler Characteristics

In this section we present a method of computing the Euler characteristic of a finite category and look to how additional category theory can aid our ability to uncover and compute Euler characteristics. We start with the notion of weightings which generalise the usefulness of Möbius invertibility to categories which do not have an explicit inversion but may have Euler characteristic regardless.

3.1 Weightings

We start by introducing an important function known as a **weighting** as follows:

Definition 3.1 (Weightings [Lei08]). Let $\mathbb{A}$ be a finite category. A **weighting** on $\mathbb{A}$ is a function $k^\bullet : \mathrm{Ob}(\mathbb{A}) \to \mathbb{Q} : \forall a \in \mathbb{A}$,

$$\sum_b \zeta(a,b)k^b = 1$$

where $\zeta(a,b) = |\mathbb{A}(a,b)|$ and $k^\bullet$ denotes the 'weight' of $\bullet \in \mathrm{Ob}(\mathbb{A})$.

We will also want to discuss what we mean by a **coweighting** but must first introduce what is meant by a 'dual' or 'opposite' category.

Definition 3.2 (Dual Categories [Lei14, p. 16]). Let $\mathbb{A}$ be a finite category. We denote its opposite or dual category by writing $\mathbb{A}^{\mathrm{op}}$. Formally, $\mathrm{Ob}(\mathbb{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathbb{A})$ and $\mathbb{A}^{\mathrm{op}}(b,a) = \mathbb{A}(a,b)\ \forall a,b \in \mathrm{Ob}(\mathbb{A})$. Composition is the same in $\mathbb{A}^{\mathrm{op}}$ as in $\mathbb{A}$ but with the arguments reversed. Explicitly, if

$$a \xrightarrow{\theta} b \xrightarrow{\varphi} c$$

are maps in $\mathbb{A}^{\mathrm{op}}$ then

$$a \xleftarrow{\theta} b \xleftarrow{\varphi} c$$

are maps in $\mathbb{A}$; these give rise to a map $a \xleftarrow{\theta\circ\varphi} c$ in $\mathbb{A}$, and the composite of the original pair of maps is the corresponding map $a \to c$ in $\mathbb{A}^{\mathrm{op}}$.

Returning to weightings, we exchange the superscript $\bullet$ with a subscript one and work over a dual category in order to define a coweighting as follows:

Definition 3.3 (Coweighting). A coweighting $k_\bullet$ on $\mathbb{A}$ is a weighting (Definition 3.1) on $\mathbb{A}^{\mathrm{op}}$.

Lemma 3.4. *$\mathbb{A}$ has Möbius inversion $\iff$ $\mathbb{A}$ has a unique weighting $\iff$ $\mathbb{A}$ has a unique coweighting; they are given by:*

$$k^a = \sum_b \mu(a,b), \quad k_b = \sum_a \mu(a,b)$$

Proof. A unique weighting, or coweighting, implies there exists some invertible function for all $a \in \mathrm{Ob}(\mathbb{A})$ which is exactly that which we have defined to be Möbius function for a finite category. Then, by definition, the Möbius function inverts the zeta function and hence by taking $k^a = \sum_b \mu(a,b)$ and $k_b = \sum_b \mu(a,b)$, we satisfy the equations $\sum_b \zeta(a,b)k^a = 1$ and $\sum_b \zeta(a,b)k_b = 1$ defining weighting and coweighting respectively. □

We now present additional theory to build on what has already been established and justify its introduction in Lemma 3.7.

Definition 3.5 (Equivalence)**.** Let $\mathbb{A}$, $\mathbb{B}$ be two categories with a functor F (Definition 2.17) between them. We say they are 'equivalent categories' if and only if both:

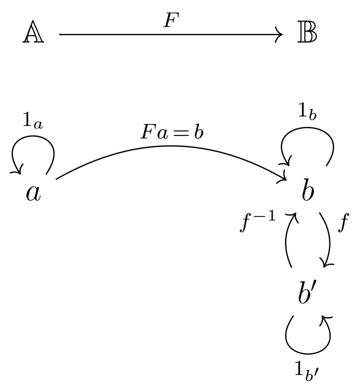
1. F is 'fully-faithful', i.e.

$$F : \mathbb{A}(a, a') \to \mathbb{A}(Fa, Fa')$$

   is an isomorphism.

2. F is 'essentially surjective', that is, each $b \in \mathbb{B}$ is isomorphic (Definition 2.7) to some Fa.

Example(s) 3.6. Consider the following categories, $\mathbb{A}$ and $\mathbb{B}$, with functor F between them:



By this construction we observe an equivalence relation on objects of $\mathbb{B}$ in which $b \sim b' \iff b, b' \cong Fa$ for $a \in \mathbb{A}$. As a has only arrows to itself and b, we observe F is fully faithful as $\mathbb{A}(a,a) = \{1_a\} \cong \{1_b = F1_a\} = \mathbb{B}(Fa, Fa)$. Then by construction, F is essentially surjective as $b' \cong b = Fa$ and we have shown there is an equivalence between $\mathbb{A}$ and $\mathbb{B}$ as a result.

Lemma 3.7. *Let $\mathbb{A}$ and $\mathbb{B}$ be equivalent finite categories (Definition 3.5) Then $\mathbb{A}$ admits a weighting $\iff$ $\mathbb{B}$ does.*

Proof. Choose an equivalence: $\mathbb{A} \xrightarrow{F} \mathbb{B}$. Let $l^\bullet$ be a weighting on $\mathbb{B}$. Then we define the weighting for some $a \in \mathbb{A}$ as follows

$$k^a = \left(\frac{C_{Fa}}{C_a} l^{Fa}\right),$$

with $C_a = \#\{a' \in \mathrm{Ob}(\mathbb{A}) : a \cong a'\}$ and similarly for $b \in \mathbb{B} : C_b = \#\{b' \in \mathrm{Ob}(\mathbb{B}) : b \cong b'\}$, denoting the number of objects in the isomorphism class of a and b respectively.

Proceed by manipulating a possible weighting on $\mathbb{A}$ (Definition 3.1). Let $a' \in \mathbb{A}$. Then $\zeta(a, a')$ and $k^{a'}$ depend only on the isomorphism class of a' $(C_{a'})$ as defined. So,

$$\sum_{a' \in \mathbb{A}} \zeta(a, a') k^{a'} = \sum_{a' \in \mathbb{A}} \zeta(Fa, Fa') k^{a'} \tag{7}$$

$$= \sum_{a' \in \mathbb{A}} \zeta(Fa, Fa') \left(\frac{C_{Fa'}}{C_{a'}} l^{Fa'}\right) \tag{8}$$

$$= \sum_{b : b \cong Fa'} \zeta(Fa, b) \frac{C_b}{C_{a'}} l^b \tag{9}$$

$$= \sum_{b \in \mathbb{B}} \zeta(Fa, b) l^b \tag{10}$$

$$= 1. \tag{11}$$

We take the initial sum that defines a weighting on an object $a' \in \mathbb{A}$. By definition, as F is an equivalence, we use that it is fully faithful so that $\zeta(a, a') = \zeta(Fa, Fa')$ in line (7). In the proceeding line (8) we can then easily substitute our above definition for $k^{a'}$. Given that F is an equivalence, F must also be essentially surjective, and so we can replace Fa' by b in line (9) as long as we begin to act over the isomorphism classes of b. However, as $a' \cong a''$ in $\mathbb{A} \iff Fa' \cong Fa'' \in \mathbb{B}$ and $C_b = C_{Fa} = \#\{b \in \mathrm{Ob}(\mathbb{B}) : b \cong Fa\}$ we note that $C_{a'} = C_b$ and thus simplify the fraction. In addition, we consider our summation to be over exactly those same $b \in \mathbb{B}$ as is presented in line (10). Then we have found that $\sum_{a' \in \mathbb{A}} \zeta(a, a') k^{a'} = \sum_{b \in \mathbb{B}} \zeta(Fa, b) l^b$.

Therefore we have established that between equivalent categories, $k^\bullet$ is a weighting on $\mathbb{A}$ provided $l^\bullet$ is a weighting on $\mathbb{B}$, as required. □

We will now present a number of categories and compute their weightings.

Example(s) 3.8. Here are a few examples constructed from familiar categories.

1. Let $\mathbb{D}_n$ be the finite discrete category of n objects such that $\zeta(a, a') = \delta(a, a')$ by definition. Then $\mathbb{D}_n$ has unique weighting as follows

$$1 = \sum_b \zeta(a, b)k^b = \sum_b \delta(a, b)k^b = k^a.$$

Therefore each $k^\bullet = 1 \; \forall \bullet \in \mathrm{Ob}(\mathbb{D}_n)$, hence $\sum_{\bullet \in \mathbb{A}} k^\bullet = |\mathrm{Ob}(\mathbb{A})| = n$.

2. Let G be a finite group, with $\mathbb{G}$ the corresponding category with unique object $\bullet$. Then $\mathbb{G}$ has unique weighting

$$1 = \sum_b \zeta(a, b)k^b = \zeta(a, a)k^a = |G| \cdot k^a$$

as $\zeta(a, a) = |G|$ by definition, we then have $|G|k^a = 1$ and hence $k^a = \frac{1}{|G|}$ (as expected, see page 5).

3. Let us take the category, $\mathbb{A}$, of the $n = 2$ poset from Example 2.3-(2). We define its weighting to be $1 = \sum_b \zeta(a, b)k^b$ for objects $a, b \in \mathbb{A}$. Computing explicitly, we write:

$$\begin{aligned}
a = \emptyset : k^{\emptyset} + \quad & k^{\{1\}} + \quad k^{\{2\}} + \quad & k^{\{1,2\}} = 1 \\
a = \{1\} : \quad & k^{\{1\}} + \quad & k^{\{1,2\}} = 1 \\
a = \{2\} : \quad & \quad k^{\{2\}} + \quad & k^{\{1,2\}} = 1 \\
a = \{1, 2\} : \quad & \quad & k^{\{1,2\}} = 1
\end{aligned}$$

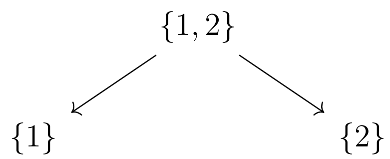
Alternatively, we may express it in matrix form as $M = (\zeta(a, b))_{a,b \in \mathbb{A}}$, i.e.

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and compute the weighting of each $\bullet \in \mathrm{Ob}(\mathbb{A})$ by taking $M * ((k^\bullet) \times (\mathbf{1})) = (\mathbf{1})$ where $(\mathbf{1})$ denotes a column vector of appropriate dimension (here it would be 4) that has all elements equal to 1. Then this forms a matrix system that can be easily solved. Consequently, we find that $k^{\emptyset} = k^{\{1\}} + k^{\{2\}} = 0$ and $k^{\{1,2\}} = 1$.

4. Categories with terminal objects have weighting $\delta(-,1)$. Hence the above example would have terminal object $\{1,2\}$ and so $k^{\emptyset} = k^{\{1\}} + k^{\{2\}} = 0$ and $k^{\{1,2\}} = 1$, as before.

5. Let $\mathbb{A}$ be the poset of non-empty subsets of $\{1,\ldots,n\}$ with reverse order, i.e. for $n = 2$



We define the weighting to be $k^J = (-1)^{|J|-1}$ as

$$\begin{aligned}
\sum_J \zeta(I,J)k^J &= \sum_{\emptyset \neq J \subset I} k^J \\
&= \sum_{\emptyset \neq J \subset I} k^{|J|-1} \\
&= \sum_{m=1}^{|I|} \binom{|I|}{m} (-1)^{m-1} \\
&= \binom{|I|}{1} - \binom{|I|}{2} + \binom{|I|}{3} + \cdots + (-1)^{|I|-1}\binom{|I|}{|I|} \\
&= \binom{|I|}{0} - \binom{|I|}{0} + \sum_{m=1}^{|I|} \binom{|I|}{m} (-1)^{m-1} \\
&= 1 - (1-1)^{|I|} \\
&= 1.
\end{aligned}$$

The formulation of a category as a matrix with entries counting arrows between objects is particularly useful in automating the computation of weightings and coweightings (see Appendix A.1).

3.2 Euler Characteristics

We lift the following Lemma (with subsequent proof) and Definition directly from [Lei08].

Lemma 3.9. *Let $\mathbb{A}$ be a finite category with weightings and coweightings $k^\bullet$ and $k_\bullet$ respectively. Then $\sum_a k^a = \sum_a k_a$.*

Proof.

$$\sum_b k^b = \sum_b \left(\sum_a k_a \zeta(a,b)\right) k^b = \sum_a k_a \left(\sum_b \zeta(a,b) k^b\right) = \sum_a k_a \qquad \square$$

Definition 3.10 (Euler Characteristic)**.** A finite category $\mathbb{A}$ has Euler characteristic if it admits both a weighting and coweighting. Its Euler characteristic is then

$$\chi(\mathbb{A}) = \sum_a k^a = \sum_a k_a \in \mathbb{Q}$$

for any weighting $k^\bullet$ and coweighting $k_\bullet$.

Example(s) 3.11. Working from the categories present in Examples 3.8, we can now remark on their Euler characteristics.

1. Let $\mathbb{D}_n$ be the finite discrete category of n objects. Then $\chi(\mathbb{D}_n) = n$.
2. Let G be a finite group of size $|G|$ with $\mathbb{G}$ the corresponding finite category with unique object $\bullet$. Then $\chi(\mathbb{G}) = \frac{1}{|G|}$.
3. Let $\mathbb{A}$ be the category of the frequently called upon $n = 2$ poset. Then $\chi(\mathbb{A}) = 1$.
4. Let $\mathbb{A}$ be a category with an initial or terminal object. As the dual of a terminal is the initial, we note that the weighting of a terminal is the same as the coweighting of an initial. Hence $\chi(\mathbb{A}) = 1$.

The process of computing Euler characteristics using the sums of weightings and coweightings is automated in Appendix A.2.

Remark 3.12. We note

$$\chi(\mathbb{A}) = \chi(\mathbb{A}^{\mathrm{op}})$$

when either side is defined.

While this is sufficient for us to compute and understand what an Euler characteristic of a category is, it would be a shame to stop there. As before, we look to existing category theory to seek out how deeply a property holds.

We continue the search much as we have before, though not explicitly called out; in searching for similar categories. We eluded to the strongest type of similarity (an isomorphism of categories) when we first spoke of skeletal categories (Definition 2.8), considering categories to be essentially isomorphic when there were no ‘redundant isomorphisms’. This was rather practical for our use case that rather quickly set the precedent that we can chip away at some of the differences present between two like-categories and still find useful results out of what remains. Moving further, we defined this weaker condition of similarity more generally as an equivalence of categories (in fact, that of 'weak' equivalence in Definition 3.5). In doing so, we preserved a lot of the underlying structure between each category, really only losing the precise number of isomorphic copies that each object has, which turned out to not be terribly important when it came to our given use case.

Finally, we come to speak on adjunctions, which are looser forms of similarity yet again, and to quote [Lan78, p.vii] 'adjoint functors arise everywhere'. Fortunately for us then, we can extend this theory of Euler characteristics to adjunctions!

The following definition is offered to provide a sense of what an adjunction is and to illustrate its notation.

Definition 3.13 (Adjunction)**.** Let $\mathbb{A}$ and $\mathbb{B}$ be finite categories. An adjunction is a pair of functors (Definition 2.17) F, G:

$$\mathbb{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{B}$$

such that there is a natural (Definition 3.14) isomorphism, i.e.

$$\mathbb{B}(Fa, b) \cong \mathbb{A}(a, Gb) \quad \forall a \in \mathbb{A}, b \in \mathbb{B}.$$

We say F is the 'left-adjoint' and G the 'right-adjoint' respectively; often writing $F \dashv G$ (with the tail directed to the left adjoint) to denote such an adjunction.

This definition in particular requires that we call on the use of the term 'naturally'; which we define below.

Definition 3.14 (Naturality)**.** Let $\mathbb{A}$ and $\mathbb{B}$ be finite categories with corresponding functors F and G between them respectively. Given an arrow $\varphi : a \to a'$ then the isomorphism $\mathbb{B}(Fa, b) \cong \mathbb{A}(a, Gb)$ is natural if the diagram

$$\begin{array}{ccc} \mathbb{B}(Fa, b) & \longleftrightarrow & \mathbb{A}(a, Gb) \\ {\scriptstyle \theta \mapsto \theta \circ F\varphi} \uparrow & & \uparrow {\scriptstyle \theta \mapsto \theta \circ \varphi} \\ \mathbb{B}(Fa', b) & \longleftrightarrow & \mathbb{A}(a', Gb) \end{array}$$

commutes for all $\theta : a \to a'$, as in:

$$Fa \xrightarrow{F\varphi} Fa' \xrightarrow{\theta} b, \qquad \theta \circ F\varphi : Fa \to b$$

and similarly for arrows constructed from $b \to b'$.

Adjunctions can be defined technically in other ways, for example, by making use of units, η, and co-units, ϵ, though we use the above formulation as it speaks most closely to the language presented already. Interested readers may seek any of the following references as good guides to uncover such alternate technical definitions. See [Lan78], [Sim11] or [Lei14].

Example(s) 3.15. Several examples of adjunctions follow.

1. (Equivalence) An equivalence (Definition 3.5) is an adjunction. Suppose $f : \mathbb{A} \to \mathbb{B}$ is an equivalence. Then given $b \in \mathbb{B}$ choose $Gb \in \mathbb{A}$ such that $FGb \cong b$ (which can always be done as F is essentially surjective).

2. (Posets) Let us take X, Y to be two finite sets with map f between them. We define 2^X to be the 'power set' of X, that is, the set of all subsets of X. Then a map $f : X \to Y$ induces an adjunction between the power sets

$$2^X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 2^Y \quad (\perp)$$

where

$$f(A) = \{f(x) \mid x \in A\}$$

for $A \in 2^X$ and

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

for $B \in 2^Y$, so that $f(A) \subset B \iff A \subset f^{-1}(B)$. Clearly then $2^X(f(A), B)$ has one element

$$\begin{aligned} &\iff f(A) \subset B \\ &\iff A \subset f^{-1}(B) \\ &\iff 2^Y\left(A, f^{-1}(B)\right) \text{ has one element.} \end{aligned}$$

Given that 2^X is a poset by inclusion, we can consider the ordered posets as described in Example 2.3-(2) to be appropriate targets for such an adjunction to exist.

3. (Open Topology) Let X and Y be finite topological spaces and $f : X \to Y$ be a continuous and open map. So that $f(U)$ is open for all open $U \subset X$ and similarly, $f^{-1}(V)$ is open for all open $V \subset Y$.

$$\mathrm{Open}(X) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \mathrm{Open}(Y) \quad (\perp)$$

where $\mathrm{Open}(X) \subset 2^X$ is a sub-poset of open sets of X. This is enough to say that we have an adjunction as $f(U) \subset V \iff U \subset f^{-1}(V)$ for all subsets of each space.

4. (Closed Topology) Let X and Y be finite topological spaces with a continuous map $f : X \to Y$ between them. Then we have an adjunction

$$\mathrm{Close}(X) \underset{f^{-1}}{\overset{\bar{f}}{\rightleftarrows}} \mathrm{Close}(Y) \quad (\perp)$$

where $\mathrm{Close}(X) \subset 2^X$ is a sub-poset of closed sets. Here we define $\bar{f}(A) = \overline{f(A)}$ for closed subsets $A \subset X$. Then we have

$$\begin{aligned} A \subset f^{-1}(B) &\iff f(A) \subset B \\ &\iff \overline{f(A)} \subset B \\ &\iff \bar{f}(A) \subset B \end{aligned}$$

as B is a closed subset of Y. Any increasing map $f : X \to Y$ between finite preorders induces an adjunction as above.

Proposition 3.16. *Let $\mathbb{A}$ and $\mathbb{B}$ be finite categories.*

a. If there is an adjunction $\mathbb{A} \rightleftarrows \mathbb{B}$ and both $\mathbb{A}$ and $\mathbb{B}$ have Euler characteristics that are defined, then $\chi(\mathbb{A}) = \chi(\mathbb{B})$.

b. If $\mathbb{A} \simeq \mathbb{B}$ then $\mathbb{A}$ has Euler characteristic $\iff$ $\mathbb{B}$ does, and in that case $\chi(\mathbb{A}) = \chi(\mathbb{B})$.

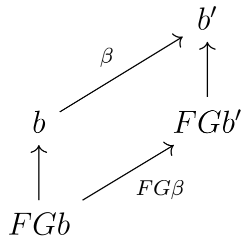
Proof.

a. Suppose there exists an adjunction $F \dashv G$ between $\mathbb{A}$ and $\mathbb{B}$. Then $\zeta(a, Gb) = \zeta(Fa, b)\,\forall a \in \mathbb{A}$ and $b \in \mathbb{B}$. We have, much like in the proof of Lemma 3.7,

$$\begin{aligned}
\chi(\mathbb{A}) &= \sum_{a\in\mathbb{A}} k_a \\
&= \sum_{a\in\mathbb{A}} \left(\sum_{b\in\mathbb{B}} \zeta(Fa, b) k^b \right) k_a \\
&= \sum_{b\in\mathbb{B}} \left(\sum_{a\in\mathbb{A}} k_a \zeta(Fa, b) \right) k^b = \sum_{b\in\mathbb{B}} \left(\sum_{a\in\mathbb{A}} k_a \zeta(a, Gb) \right) k^b \\
&= \sum_{b\in\mathbb{B}} k^b \\
&= \chi(\mathbb{B})\,.
\end{aligned}$$

Hence, if we know $\mathbb{A}$ and $\mathbb{B}$ to have Euler characteristics, they must be equal.

b. Recall that any equivalence is an adjunction. To see this suppose $F : \mathbb{A} \to \mathbb{B}$ is an equivalence. For each $b \in \mathbb{B}$ we choose an $a \in \mathbb{A}$ such that $b \cong Fa$. When $b = Fa$ we choose $Gb = a$. Set $Gb = a$ so that $G\beta \in \mathbb{A}(Gb, Gb')$ is the unique arrow such that $FG\beta$ is composite, namely:



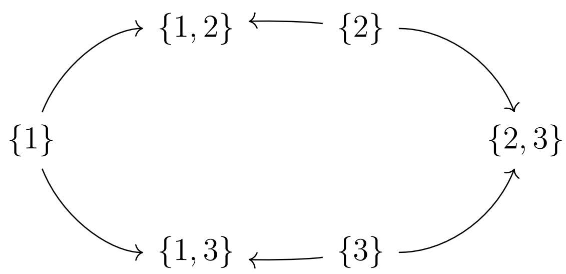
This defines our functor $G : \mathbb{B} \to \mathbb{A}$. Hence $\mathbb{A}(a, Gb) \cong \mathbb{B}(Fa, b)$. Then $\mathbb{A}$ has $\chi \iff \mathbb{B}$ does by Lemma 3.7. Hence if $\mathbb{A}$ has χ then $\chi(\mathbb{A}) = \chi(\mathbb{B})$ by part (a). □

Example(s) 3.17. Computing the Euler characteristic via adjunctions (Examples 3.15).

1. $\chi\big(2^X\big) = 1$ for any finite set X and as we have initial or terminal objects.

2. Both $\chi(\mathrm{Open}(X)) = 1$ and $\chi(\mathrm{Close}(X)) = 1$ as we have initial and terminal objects.

3. (Spherical poset) Let us take a poset 2^X for $X = \{1, 2, 3\}$ and take out its initial and terminal objects. Then we construct the diagram of $2^X \setminus \{\emptyset, X\}$ in line with the spherical moniker as follows:

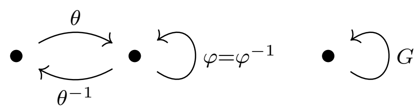


Then we can compute its Euler Characteristic to be $\chi\big(2^X \setminus \{\emptyset, X\}\big) = 0$. More generally χ can be computed based on the parity of the set cardinality:

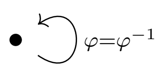
$$\chi\big(2^X \setminus \{\emptyset, X\}\big) = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ 2 & \text{if } |X| \text{ even.} \end{cases}$$

Similarly, the spherical poset for $X = \{1, 2\}$ is just the discrete category of 2 objects, hence as in Example 3.11-(1) it has $\chi = 2$.

4. (Groupoid) Let $\mathbb{P}$ be a finite groupoid (i.e. all arrows are isomorphisms), of the form:



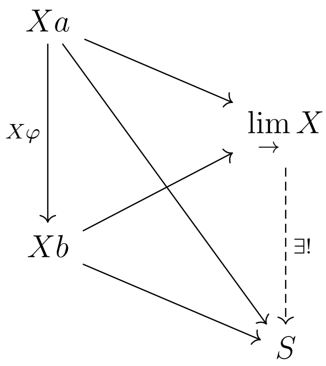
Then we write $\mathbb{A} \simeq \sum_i^2 \mathbb{P}_i$, splitting the groupoid into the connected-components of automorphism groups. In this example we have:

$\mathbb{P}_1 =$ 

and $\mathbb{P}_2 = \mathbb{G}$ (just the finite group category). So that $\chi(\mathbb{P}) = \chi(\mathbb{A}) = \chi(\sum \mathbb{P}_i) = \frac{1}{2} + \frac{1}{|G|}$. In general we have $\mathbb{A} \simeq \sum_i \mathbb{P}_i$ and $\chi(\mathbb{A}) = \sum_i \frac{1}{|\mathbb{P}_i|}$.

4 Colimits

Definition 4.1. Let $\mathbb{A}$ be a finite category and X a functor as in $X : \mathbb{A} \to$ **Set**. Thus Xa is a finite set and for $\varphi \in \mathbb{A}(a, b)$ we have a map $X\varphi : Xa \to Xb$. The colimit, denoted $\varinjlim X$, is the set such that there exists a map $Xa \to \varinjlim X, \forall a \in \mathrm{Ob}(\mathbb{A})$ compatible with all $X\varphi$ and such that the following diagram naturally commutes for all φ when S is any other set with these properties.

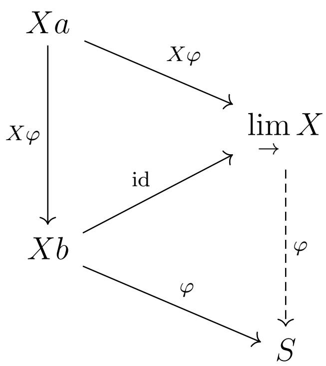


This last 'universal' property uniquely determines S up to bijection.

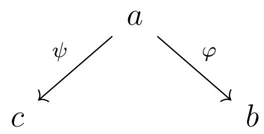
Example(s) 4.2.

1. Let $\mathbb{A}$ be a category consisting of two distinct objects. Then $\varinjlim X = Xa + Xb$; representing their disjoint union.

2. Let $\mathbb{A}$ be a category consisting of two objects and a single arrow between them $\varphi : a \to b$. Then $\varinjlim X = Xb$ because the following diagram commutes



3. Let us consider a 'pushout' diagram

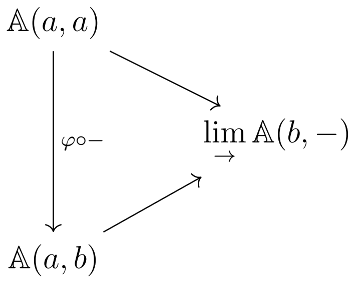


for which $X\varphi$ and $X\psi$ are injections. Then $\varinjlim X = Xb \bigcup_{Xa} Xc$ is the union of Xb and Xc with elements of the common subset Xa identified.

4. Consider a finite group G with $\mathbb{G}$ the corresponding category of G consisting of one object $\bullet$ and $\mathbb{G}(\bullet, \bullet) = G$. We have that the functor $X : \mathbb{G} \to \textbf{Set}$ is a G-action on the set $X(\bullet)$ because we have bijections $Xg : X(\bullet) \to X(\bullet)$ for each $g \in G$. This functor does not demand then that group actions are free, i.e. Xg can have fixed points. The representable functor $X = \mathbb{G}(\bullet, -)$ with $X(\bullet) = \mathbb{G}(\bullet, \bullet) = G$ corresponds to the free action $G \times G \to G$ of G on itself. Free G actions correspond to familially representable functors $X = n\mathbb{G}(\bullet, -)$ where n is the number of orbits. In all cases $\varinjlim X = X(\bullet)/G$ is the quotient.

Proposition 4.3. *Let $\mathbb{A}$ be a finite category and $k^\bullet$ a weighting on $\mathbb{A}$. If $X : \mathbb{A} \to \textbf{Set}$ is finite and familially representable (Definition 2.18) then* $|\varinjlim X| = \sum_a k^a |Xa|$

Proof. We begin by showing $|\varinjlim \mathbb{A}(a, -)| = 1 \ \forall a \in \mathbb{A}$. Let $\bullet \in \varinjlim \mathbb{A}(a, -)$ be the image of $\mathrm{id}_a \in \mathbb{A}(a, a)$ under the map $\mathbb{A}(a, a) \to \varinjlim \mathbb{A}(a, -)$. Then given any arrow $\varphi \in \mathbb{A}(a, b)$ we see φ also maps to $\bullet$ under $\mathbb{A}(a, b) \to \varinjlim \mathbb{A}(a, -)$ as the following diagram:



commutes. This shows that the image of φ in $\varinjlim \mathbb{A}(a, -)$ is also the image of id_a and so via the universal property we have $\varinjlim \mathbb{A}(a, -) \cong \{1_a\}$, a one element set. Hence $|\varinjlim \mathbb{A}(a, -)| = 1$.

Second note that for $X = \mathbb{A}(a, -)$ we likewise have

$$\sum_b k^b |Xb| = \sum_b k^b |\mathbb{A}(a, b)| = \sum_b k^b \zeta(a, b) = 1$$

by Definition 3.1 of weightings. Therefore the result is true for a representable functor $X = \mathbb{A}(a, -)$. The result follows for familially representable functors because both sides are additive. If X and Y are finite functors, then $(X + Y)a = Xa + Ya$ (is the disjoint union of sets). So that

$$\lim_{\rightarrow} (X + Y) = \lim_{\rightarrow} X + \lim_{\rightarrow} Y$$
$$\text{and } |\lim_{\rightarrow} (X + Y)| = |\lim_{\rightarrow} X| + |\lim_{\rightarrow} Y|.$$

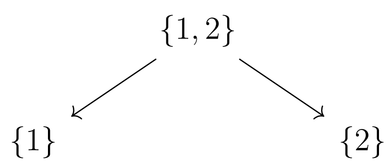
Similarly, we can split the disjoint union by use of the weighting as follows:

$$\begin{aligned} \sum_b k^b |(X + Y)b| &= \sum_b k^b |Xb + Yb| \\ &= \sum_b k^b (|Xb| + |Yb|) \\ &= \sum_b k^b |Xb| + \sum_b k^b |Yb|. \end{aligned}$$

Hence, any familially representable $X : \mathbb{A} \to \mathbf{Set}$ has $|\lim_{\rightarrow} X| = \sum_b k^b |Xb|$, as expected. $\square$

Example(s) 4.4.

1. Let $\mathbb{A}$ be the opposite of the poset of non-empty subsets of $\{1, \ldots, n\}$. So for $n = 2$, it is the category



   and so on. Let X be a finite set and $X_1, \ldots, X_n$ subsets of X. Define a functor $X : \mathbb{A} \to \mathbf{Set}$ (which **is** familially representable, though this is not obvious) by $X(I) = \bigcap_{i \in I} X_i$ so that $X(\{k\}) = X_k$ etc. and take the arrow $I \to J$ to the inclusion

$$X(I) = \bigcap_{i \in I} X_i \subset \bigcap_{j \in J} X_j = X(J).$$

Then $\lim\limits_{\to} X = \bigcup_{i=1}^{n} X_i$. So by Proposition 4.3 and Example 3.8-(5), we obtain

$$|\cup_{i=1}^{n} X_i| = |\lim_{\to} X| = \sum_{I \in \mathbb{A}} k^I |X(I)| = \sum_{I} (-1)^{|I|-1} |\cap_{i \in I} X_i| \,.$$

Which is just the Principle of Inclusion-Exclusion.

2. A finite group G acts freely on a set S. Let $\mathbb{G}$ be the corresponding category of G consisting of one object $\bullet$ and $G(\bullet, \bullet) = G$. Let $X : \mathbb{G} \to$ **Set** be a functor $X(\bullet) = S$ (which is familially representable $\iff$ action is free) and $X(g) = g : S \to S$ the map defined by actions on G. Then $\lim\limits_{\to} X = S/G$ and, once more, by Proposition 4.3 we obtain

$$|S/G| = |\lim_{\to} X| = \sum_{a \in \mathbb{G}} k^a |X(a)| = \frac{1}{|G|} |S|.$$

A Maple Program

A.1 Computing the (co)weightings of a finite category

```
Weightings := proc(ExampleMatrix::Matrix, {coweight::boolean := false})
global free;
local ProcMatrix, RDim, OnesMatrix, RCT;
# Check if user desires the coweight, if so, convert. Else continue with input.
# The check is done here so that we can precisely check the conditions which fail (if any) for a weighting or coweighting separately.
    if coweight then
        ProcMatrix := transpose(ExampleMatrix);
    else
        ProcMatrix := ExampleMatrix;
    end if;
# We establish this as a variable as we call on it again if it passes the next check.
    RDim := RowDimension(ProcMatrix);
# Check to see if the matrix has been input properly
    if RDim <> ColumnDimension(ProcMatrix) then
        error "You did not input an n x n matrix as expected.";
    end if;
# Knowing that the input matrix is (n x n), we can now construct a
# corresponding (n x 1) matrix full of 1s
    OnesMatrix := Matrix(RDim,1,1);
# Check for rank difference to determine inconsistent solutions (=> no (co) weighting)
# by Rouche-Capelli Theorem; hence the RCT variable.
    RCT := Rank(ProcMatrix) < Rank(<ProcMatrix|OnesMatrix>);
# If we asked for the coweight and it happens that the rank is either equal or greater
# than the augmented then we have no solution.
    if coweight and RCT then
        error "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no coweighting.";
    elif RCT then
       error "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no weighting.";
    end if;
# If we passed through all of these checks, then our solutions are unique or there are infinitely many.
# So when returning the solutions, if there are infinitely many we ask the free variable to be in x for legibility.
    return Linearsolve(ProcMatrix, OnesMatrix, free='x');
end proc:
```

Listing 1: Weightings Procedure

A.1.1 Code Commentary

The `Weightings` procedure has one mandatory argument (an input matrix) and an optional flag 'coweight' to indicate that you are interested in the coweighting of a category. If this optional argument is given, we proceed by transposing the matrix which gives us the coweight matrix without the need to input that separately ourselves. Following this we build some local variables based on the matrix we are proceeding with to assess whether it has been constructed correctly and whether it will have solutions (determined by the conditions of rank by the Rouché–Capelli theorem). If the correct conditions are not met then we check in the final `if` loop whether we were working on the weighting or coweighting matrix and exit the program with an error code explaining why there was no solution. Otherwise, should the conditions for a solution exist we can continue and return an output, accepting infinite solutions with a free variable in x.

A.1.2 Example input/output

```
> ExampleFreeVar := Matrix([[1,1],[1,1]])
```

$$ExampleFreeVar := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (1)$$

```
> Weightings(ExampleFreeVar)
```

$$\begin{bmatrix} 1 - x_{1,1} \\ x_{1,1} \end{bmatrix} \quad (2)$$

```
> ExampleNoCW := Matrix([[1,1,2],[1,1,2], [1,2,1]])
```

$$ExampleNoCW := \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad (3)$$

```
> Weightings(ExampleNoCW)
```

$$\begin{bmatrix} 1 - 3\,x_{1,1} \\ x_{1,1} \\ x_{1,1} \end{bmatrix} \quad (4)$$

```
> Weightings(ExampleNoCW, coweight)
Error, (in Weightings) The rank of the augmented matrix is greater than the coefficient's rank, hence there is no coweighting.
> ExampleFracEC := Matrix([[1,1,2],[2,1,1],[1,2,1]])
```

$$ExampleFracEC := \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad (5)$$

```
> Weightings(ExampleFracEC)
```

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \quad (6)$$

A.2 Computing $\chi(\mathbb{A})$ of a finite category

```
EulerChar := proc(Examplematrix::Matrix);
global coweight, chi;
local W, CW;
# We recall the weightings procedure output as a column vector, and take the sum of its components.
    try
        (W := sum(Weightings(ExampleMatrix))
    catch
        "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no weighting.":
   end try;
    try
       (CW := sum(Weightings(ExampleMatrix, coweight)))
    catch
       "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no coweighting.":
   end try;
# If their sum agrees then we say so and print the euler characteristic,
# else we admit otherwise and print their weightings to display their differences.
    if (W=CW) then
        print("The weightings agree");
        print(chi = W);
    else
        print("The weightings disagree");
        print(W, CW);
    end if;
end proc:
```

Listing 2: Euler Characteristic Procedure

A.2.1 Code Commentary

The `EulerChar` procedure takes in a sole input matrix and feeds it to the `Weighting` procedure defined above twice in order to check that there exists both weighting and coweighting. We use `try` and `catch` to allow the procedure to continue despite the fact the `Weighting` procedure has failed (which would otherwise also terminate this procedure at this step). If a set of weightings or coweightings exists then it takes their sum. Later, if both exist and are equal, we deduce that this is the Euler characteristic and so we output that they agree and choose arbitrarily between them to output $\chi =$ `W`. If they are not equal, then we output their sums when they exist (as these are what we attempted to store as `W`, `CW`) and claim that the weightings disagree. In the case where, for example, a weighting exists but not a coweighting, the procedure shall output what it calculated to be the sum of the weighting and `CW` in a list; indicating that the latter had the incompatible system of equations. This case can be seen in the next subsection for `ExampleNoCW`.

One could point out inefficiencies in the design of this procedure as it has to call on `weightings` twice. Admittedly, it would seem exceptionally likely that either procedure could be restructured to cut down on computation time, though at present, with all examples tested the program does run acceptably fast; and far faster than any manual computation could hope to achieve.

A.2.2 Example input/output

```
> EulerChar(ExampleFreeVar)
                    "The weightings agree."
                           χ = 1                              (1)

> EulerChar(ExampleNoCW)
                    "The weightings disagree."
                       1 - x[1, 1], CW                        (2)

> EulerChar(ExampleFracEC)
                    "The weightings agree."
                           χ = 3/4                            (3)
```

B Statement of Originality

This dissertation was written by me, in my own words, except for quotations from published and unpublished sources which are clearly indicated and acknowledged as such. I am conscious that the incorporation of material from other works or a paraphrase of such material without acknowledgement will be treated as plagiarism, according to the University Academic Integrity Policy. The source of any picture, map or other illustration is also indicated, as is the source, published or unpublished, of any material not resulting from my own research.

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