MATH554:
Series Euler Characteristic

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Contents

1 Introduction

In this dissertation we wish to reconstruct the background required to understand the construction of the 'series Euler characteristic' (from [BL08]) in order to later compare it against the 'regular Euler characteristic' (from [Lei08]). For both constructions we shall find sensible and grounded answers that, interestingly, do not always agree[1].

We begin by recounting simplicial complexes, initially with their geometric construction and then after introduce the abstract analogue as well as how one may transition between either construction in Section 2.1.1. Later we define their Euler characteristic to be the alternating sum of p-simplices. This is a construction many will be familiar with, perhaps first as 'Euler's Polyhedron Formula'; whereby any convex polyhedron (tetrahedron, cube and so on) has Euler characteristic $\chi = \mathbf{v} - \mathbf{e} + \mathbf{f} = 2$, where the **v**ertices, **e**dges and **f**aces are simplices of order zero, one and two respectively. In addition, we prove that the Euler characteristic is invariant under barycentric subdivision.

Following this, we show how any abstract simplicial complex can be turned into as a poset of simplices and that naturally this poset then can be given the same Euler characteristic via its non-degenerate nerve (which is the barycentric subdivision of the underlying abstract simplicial complex). In Leinster's initial paper, [Lei08], the Möbius inversion of posets by Gian-Carlo Rota [Rot64] is generalised for finite categories and both give rise to a natural definition of Euler characteristic (which we consider the 'regular' Euler characteristic).

We then provide a brief recap of the regular Euler characteristic of finite categories as covered in my preliminary dissertation using the '(co)weightings' of each finite category $\mathbb{A}$. Afterwards, we explore the necessary background for the complete construction of the 'series' Euler characteristic. Beginning first with a recap of formal power series, then of delta sets (also known as 'semi-simplicial' sets) and then ultimately relating the nerve of a finite category to its classifying space to equate their Euler characteristics. In our construction of delta sets we show how they can easily represent geometric and abstract simplicial complexes. In fact, the delta set is the corresponding 'cell complex' of finitely many non-degenerate n-simplices of each dimension n. Thinking that a sensible definition of Euler characteristic may be "an alternating sum of cell complexes", one can then consider a formal power series of the form $f(t) = \sum_{n\geq 0} c_n t^n$ where c_n denotes the number of cells in each dimension (i.e. n-simplices and α_n) and evaluated at $t = -1$ to produce an agreeable definition for geometric and simplicial complexes, namely $\sum_{n\geq 0}(-1)^n \alpha_n$. We

[1] See Section 2.3.4

then reconsider the 'matrix of morphisms' that is, a matrix whose entries e_{ij} are the numbers of morphisms between the (arbitrarily) ordered objects i and j of a given finite category. We go on to show that a category with non-degenerate nerve corresponding to a delta set has a rational formal power series that converges at $t = -1$ and in turn produce an appropriate definition of a series Euler characteristic by evaluating this formal power series $f_{\mathbb{A}}$ at $t = -1$. Following this, we compute the regular and series Euler characteristic for a handful of finite categories. I then explore how the series Euler characteristic behaves well with respect to the categorical sum in Proposition 2.46 and how one can consider this as an arbitrary partitioning of the block sum of different matrices that correspond to differing finite categories than one used to initially construct the block. In doing so, we consider whether there is any way to include additional morphisms between objects in categories that preserves their series Euler characteristics. Unfortunately, this approach does not seem to bear fruit in this regard and, as it stands, seems to show that for a class of finite category (namely those that are block sums of smaller finite categories with series Euler characteristic) that there are no possible morphisms that can be added without altering series Euler characteristic.

Ultimately, we round up in Section 3 by introducing the Euler characteristic of an orbifold as given to us by John Conway and Daniel Huson [CH02] and show that the series Euler characteristic is equally sufficient to demonstrate that the 17 wallpaper groups have Euler characteristic zero. We first do so by recounting the definition of the orbifold Euler characteristic as the reduction of the Euler characteristic of a compact surface (typically a sphere) that is adjoined with holes, handles, as well as corner and cone points (known quaintly as 'defects') and considering the 17 wallpaper groups as a problem of enumerating the possible defects. Then we demonstrate in Section 3.1 that the series Euler characteristic of the matrix of morphism corresponding to an orbifold with cone point agrees with existing definitions for the Euler characteristic and that corner points follow trivially. We finish off by providing a matrix of morphism for each of the seven surfaces with integer Euler characteristic greater than or equal to 0.

In Appendix A there are a number of Maple programs that automate the computation of both regular and series Euler characteristics for a given matrix of morphisms, adjoined by sensible commentary where necessary. I reuse the same stylings file created for my previous dissertation to input aesthetically pleasing syntax-highlighted code into LaTeX as preexisting offerings were lacklustre.

Acknowledgements I thank Jon Woolf very dearly for his infectious enthusiasm and the tremendous guidance he provided on both of my

dissertations, as well as the impact he has had on my writings. I am additionally grateful to Tom Leinster for his original work that both of my dissertations follow closely.

2 Related algebraic Euler Characteristic constructions

In this section we present the necessary background for later sections and discuss the importance of results along the way. We wish to uncover a general definition of Euler characteristic that is consistent across a large number of structures. To do so, we first look to finite simplicial complexes, then later exploit their relation to the nerves of posets and finite categories to uncover each structure's respective Euler characteristic.

2.1 Finite Simplicial Complexes

We begin by introducing simplicial complexes, both geometric and abstract and discuss how we can translate between the two. Further theory and additional background can be found in section three of [Gib10] which we use extensively as reference material here.

We start by defining what a simplex is. Let us present the following definitions with preferred notation for the geometric case first.

Definition 2.1 (Geometric Simplex)**.** A **geometric simplex**, $\sigma \in \mathbb{R}^N$, is the convex hull of $n+1$ affinely independent vectors, $v_0 \ldots, v_n \in \mathbb{R}^N$. So that we write each simplex σ as the convex hull: $\sigma = \{\sum_{i=0}^{n} \lambda_i v_i \colon \sum \lambda_i = 1, \lambda_i \geq 0\}$.

Definition 2.2 (Face of a Geometric Simplex)**.** A **geometric face** of σ is a simplex given by the convex hull of a non-empty subset of the vertices $\{v_0, \ldots, v_n\}$. If $\tau \in X$ (for some set X of simplices) is a face of σ, then we write $\tau \leq \sigma$ (or equally $\sigma \geq \tau$). Note that $\sigma \leq \sigma$ for any simplex σ.

Definition 2.3 (Geometric Simplicial Complex)**.** Let $X \subset \mathbb{R}^N$ be a finite set of geometric simplices in $\mathbb{R}^N$ such that:

1. If $\sigma \in X$ and $\tau \leq \sigma$ (τ is a geometric face of σ) then $\tau \in X$, and

2. $\sigma, \tau \in X \implies \sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a face of both σ and τ, and hence $\sigma \cap \tau \in X$ by (1).

Then X is a (finite) **geometric simplicial complex**.

With this construction, it is not so hard to see 0, 1 and 2-simplices as points, lines and triangles, or, put another way: vertices, edges, and faces. Moreover, a 3-simplex is a tetrahedron and so on, though these become increasingly harder to visualise.

In contrast, we may view simplices more abstractly as follows:

Definition 2.4 (Abstract Simplex, Face and Simplicial Complex)**.** Let X be the pair (V, S) for a set V whose elements are 'vertices' and S a set of non-empty subsets of V. Elements $\sigma \in S$ are the **abstract simplices**, and for sets in S of size $n+1$ we consider those the **abstract n-simplices**. Each σ then has a **face** for each non-empty subset $\tau \colon \tau \subset \sigma \in S$, with $\sigma \subset \sigma\ \forall \sigma$, trivially. Provided we have the following:

1. $v \in V \implies \{v\} \in S$ and
2. if $\sigma \in S$ and $\emptyset \neq \tau \subset \sigma$ (τ is a face of σ) then $\tau \in S$,

then $X = (V, S)$ is an **abstract simplicial complex**.

Remark 2.5. S is contained in the non-empty subsets of the powerset 2^V.

Consequently, we observe, much like the geometric case, that any pair $\sigma, \tau \in S$ has either no intersection: $\sigma \cap \tau = \emptyset$, or that they meet at a common face, $(\sigma \cap \tau \subset \sigma) \wedge (\sigma \cap \tau \subset \tau)$.

2.1.1 Geometric to abstract and back

We spend this subsection justifying our abstraction of the geometric simplicial complex by presenting methods to translate between the two below. From here, we shall use the shorthand X_A for an abstract simplicial complex. Similarly, we refer to a geometric simplicial complex as X_G.

Geometric to abstract: We begin with our geometric simplicial complex X_G and construct an abstract simplicial complex $S_{X_G} = (V, S)$ which we shall refer to as S_{X_G}. First we take V to be the set of 0-simplices in X_G and each simplex σ is in $S_{X_G} \iff$ the convex hull: $\langle v \colon v \in \sigma \rangle$ is in the complex X_G. As before S is a subset of the non-empty subsets of V, i.e. $S \subset 2^V$, a powerset. Note that if $\sigma \in S_{X_G}$ then $\tau \subset \sigma \implies \tau \in S_{X_G}$, since faces of geometric simplices in X_G are geometric simplices in X_G.

So we conclude S_{X_G} is an abstract simplicial complex

Abstract to geometric: Conversely, given a (finite) abstract simplicial complex $X_A = (V, S)$ with $S \subset 2^V$, we can realise it as a geometric complex in $\mathbb{R}^{|V|}$ where $|V|$ denotes the cardinality of the set V. We do so by ordering vertices in V as $v_1, \ldots, v_{|V|}$ and realising $\sigma \in S$ as the convex hull

$$|\sigma| := \langle e_i \colon v_i \in \sigma \rangle$$

where each e_i is the i-th standard coordinate vector with 1 in the i-th column, and zeroes elsewhere. For example, let $|V| = 3$ and $\sigma = \{1, 3\}$ so that $|\sigma| = \langle e_1, e_3 \rangle$. In general, $S \subset 2^V$ gives us a set of geometric simplices in $\mathbb{R}^V$, with the property that both $\tau \subset \sigma \implies |\tau|$ is a face of $|\sigma|$ and $|\tau| \cap |\sigma| = |\tau \cap \sigma|$.

Therefore we may conclude that $|X_A| = \{|\sigma| \colon \sigma \in S\}$ is a geometric simplicial complex.

Having discussed the relationship between geometric and abstract simplicial complexes, we consider their isomorphisms. We begin by defining an isomorphism between geometric simplicial complexes $X_G \subset \mathbb{R}^M$ and $Y_G \subset \mathbb{R}^N$ as follows:

Definition 2.6 (Isomorphism (Geometric))**.** Let X_G, Y_G be geometric simplicial complexes of dimension M and N respectively. Then a linear map $L \colon X_G \to Y_G$ such that

1. for each simplex, $\sigma \in X_G$, $L(\sigma)$ is a simplex in Y_G of the same dimension, and

2. each simplex $\tau \in Y_G$ is of the form $\tau = L(\sigma)$ for a unique $\sigma \in X_G$.

is an **isomorphism of geometric simplicial complexes**. When there is such an isomorphism we write $X_G \cong Y_G$.

Then for the abstract simplicial complex we have

Definition 2.7 (Isomorphism (Abstract))**.** Let $X_A = (V, S)$ and $Y_A = (W, T)$ with $S \subset 2^V$ and $T \subset 2^W$ be abstract simplicial complexes. If there exists a bijection $f \colon V \to W$ such that $\sigma \in S \iff f(\sigma) \in T$, then f is an **isomorphism of abstract simplicial complexes**. When there is such an isomorphism we write $X_A \cong Y_A$.

Remark 2.8. Consider the linear map $L \colon \mathbb{R}^V \to \mathbb{R}^N$, mapping $e_i \mapsto v_i$ for each $v_i \in V$. Then this induces an isomorphism of geometric simplicial complexes, $L \colon |S_{X_G}| \cong X_G$.

Remark 2.9. Consider the map of vertex sets $f \colon V \to \{e_i \colon i = 1, \ldots, |V|\}$ given by mapping v_i to e_i. Then this induces an isomorphism of abstract simplicial complexes $f \colon X_A \cong S_{|X_A|}$.

2.1.2 Euler Characteristic

Having covered both geometric and abstract simplicial complexes we can now look to define their Euler Characteristics. We begin first with the geometric case and define it as in [Gib10, Definition 6.15].

Definition 2.10 (Euler Characteristic (Geometric Simplicial Complex))**.** Let X_G be a geometric simplicial complex of dimension n. Let $\alpha_p(X_G)$ denote the number of p-simplices in X_G. Then

$$\chi(X_G) = \sum_{p=0}^{n} (-1)^p \alpha_p(X_G), \tag{1}$$

is the **Euler Characteristic of** X_G.

We can write the formula in a different way. Let us translate the formula for the geometric Euler characteristic, $\chi(X_G)$, into a definition of $\chi(X_A)$ for an abstract simplicial complex X_A. Suppose we represent each simplex as σ, with the set of all simplices S. Then each p-simplex in X_G corresponds to a p-simplex in X_A which has cardinality equal to $p+1$ (Definition 2.4), so that the number of p-simplices in X_G, that is, $\alpha_p(X_G)$ is the sum:

$$\alpha_p(X) = \sum_{\substack{\sigma\in S \\ |\sigma|=p+1}} 1,$$

and we can rewrite the above formula (1) above as follows:

$$\begin{aligned}
\chi(X) &= \sum_{p=0}^{n} (-1)^p \alpha_p(X) \\
&= \sum_{p=0}^{n} (-1)^p \sum_{\substack{\sigma\in S \\ |\sigma|=p+1}} 1 \\
&= \sum_{p=0}^{n} \sum_{\substack{\sigma\in S \\ |\sigma|=p+1}} (-1)^{|\sigma|-1} \\
&= \sum_{\sigma\in S} (-1)^{|\sigma|-1}.
\end{aligned}$$

In doing so, we have defined a sensible formula for the Euler characteristic of an abstract simplicial complex. Hence, we write

Definition 2.11 (Euler Characteristic (Abstract Simplicial Complex))**.** Let $X_A = (V, S)$ be an abstract simplicial complex. Let $|\sigma|$ denote the cardinality of the simplex $\sigma \in S$. Then

$$\chi(X_A) = \sum_{\sigma \in S} (-1)^{|\sigma|-1},$$

is the **Euler Characteristic of** X_A.

2.1.3 Barycentric Subdivision and invariance of Euler Characteristic

Continuing our bias for presenting the geometric case first, we define the 'barycentre' for a geometric simplex as follows:

Definition 2.12 (Barycentre and Barycentric Subdivision (Geometric))**.** For each simplex $\sigma = \{v_i : i \in I\}$ in a geometric simplicial complex, X_G, we define its **barycentre** to be the point at its centroid, namely: $\hat{\sigma} = \frac{1}{|I|} \sum_{i \in I} v_i$.

The (first) **Barycentric Subdivision** of the total simplicial complex X is the set of points $\{\hat{\sigma} : \sigma \in X_G\}$ as well as all simplices τ with convex hull $\tau = \langle \hat{\sigma}_i : i = 0, \ldots, q \rangle$ where each $\sigma_{i-1} \leq \sigma_i$ for each i and each σ_i is a simplex of X_G. The barycentric subdivision of X is a geometric simplicial complex, as in Definition 2.3 and is denoted X'; with subsequent subdivisions often denoted with additional dashes.

Let us now consider what the barycentre of the first few k-simplices are and what simplicial complex looks like after subdivision.

Trivially, the barycentre of each 0-simplex is at the same point, (v). A 1-simplex has barycentre at the midpoint of its line, $\frac{1}{2}(v_i + v_j)$, and a 2-simplex, as a face, has barycentre at its centroid: $\frac{1}{3}(v_i + v_j + v_k)$. In total, we can consider the barycentric subdivision of some geometric complex in the plane to do the following:

1. Replaces each existing 0-simplex with a 0-simplex at the point itself,

2. Each 1-simplex obtains an additional 0-simplex at its midpoint and is itself replaced by two 1-simplices joining both endpoints to the new 0-simplex, and

3. Each 2-simplex produces a 0-simplex at its centroid, with additional 1-simplices joining all 0-simplices of the boundary to its centroid, forming six new 2 simplices in total.

With this intuition built for geometric barycentric subdivision, we can now consider the following for the abstract simplicial complex.

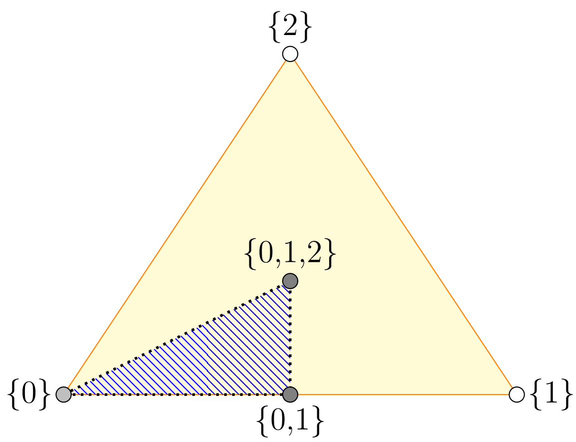
Definition 2.13 (Barycentre and Barycentric Subdivision (Abstract))**.** Let $X_A = (V, S)$ be an abstract simplicial complex, with V the set of vertices and S the non-empty subsets of the powerset 2^V with each simplex $\sigma_i \in S$. We define the **barycentric subdivision of an abstract simplicial complex** to be $X'_A = (V', S')$, where the set of vertices in the barycentric subdivision are the previous subsets of 2^V, namely: $V' = S$. The set $S' \subset 2^{V'}$ of simplices of X'_A consists of the subsets $\{\sigma_0, \dots, \sigma_p\}$ where

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_p, \quad \text{with } \sigma_i \in S \text{ for } i = 0, \dots, p$$

is a non empty chain of simplices in X'_A. Given this construction, it is easy to see that X'_A is again an abstract simplicial complex as in Definition 2.4.

Example(s) 2.14. Let us look at a 2-simplex: $\{0, 1, 2\}$. Then we we have one such chain $\{0\} \subset \{0, 1\} \subset \{0, 1, 2\}$ which presents each point in the chain as a face (Definition 2.4) of its smaller subset. Hence, we obtain a 0-simplex for each element in the chain (corresponding to an initial corner of its face, the midpoint from this corner to another and the point at its centroid respectively), and a 1-simplex between each 0-simplex, with a final 2-simplex being produced with each 0-simplex acting as its new corner. By considering all such chains we obtain an equivalent representation to the geometric case.

The following is an illustrative diagram of the above example.



Lemma 2.15.

1. *If X_G is a geometric simplicial complex with associated abstract simplicial complex S_{X_G}, then $X'_G \cong |S'_{X_G}|$.*

2. *If X_A is an abstract simplicial complex with associated geometric simplicial complex $S_{|X_G|}$, then $X'_A \cong S_{|X_G|'}$.*

Proof.

1. Suppose X is a geometric simplicial complex. Let S_X be the corresponding abstract complex. Then the simplices in X' and in S'_X correspond bijectively to chains $\sigma_0 \subset \cdots \subset \sigma_p$ of simplices in X. There is a map $X' \to |S'_X|$ taking a point $\sum_{i=0}^{p} t_i \hat{\sigma}_i$ in the convex hull $\langle \hat{\sigma}_i \colon i = 0, \ldots, p \rangle$ to the corresponding point $\sum_{i=0}^{p} t_i e_{\sigma_i}$ where e_{σ_i} is the standard basis vector corresponding to the vertex σ_i of S'_X. This defines an isomorphism $X' \cong |S'_X|$ of geometric complexes.

2. Suppose X is an abstract simplicial complex. Let $|X|$ be a corresponding geometric realisation of X. Simplices in X' and in $|X|'$ correspond bijectively to chains $\sigma_0 \subset \cdots \subset \sigma_p$ of simplices in X. There is a map $X' \to S_{|X|'}$ taking a vertex σ of X' (where σ is a simplex in X) to the vertex $\widehat{|\sigma|}$ (i.e. the barycentre of the realisation $|\sigma|$ of σ) and defined on the interior of each simplex by linear extension. This defines an isomorphism $X' \cong S_{|X|'}$ of abstract complexes. $\square$

Having established barycentric subdivision is equivalent for geometric and abstract simplicial complexes, we would now like to observe whether it has any effect on Euler characteristic. We follow the outline of a proof present in [Gib10, pp. 55] to guide the following proposition:

Proposition 2.16. *The Euler Characteristic of a 2d simplicial complex, X_G, is invariant under barycentric subdivision.*

Proof. First, we note that a 2-dimensional simplicial complex $X_G \subset \mathbb{R}^2$ has Euler characteristic: $\chi(X_G) = \alpha_0 - \alpha_1 + \alpha_2$. Writing α'_i to denote the count of each i-simplex after barycentric subdivision of X_G, we justify the impact for each:

i=0 : $\alpha'_0 = \alpha_0 + \alpha_1 + \alpha_2$. As each vertex replaces itself with one new vertex, each edge produces an additional vertex at its barycentre and similarly each face produces a vertex at its barycentre.

i=1 : $\alpha'_1 = 2\alpha_1 + 6\alpha_2$. We find each existing edge is split into two new edges and that the barycentre of each face creates six new edges per face as it connects each existing vertex to its barycentre.

i=2 : $\alpha'_2 = 6\alpha_2$. Finally, as each face was divided by six new edges to the centroid, we have produced six faces in place of each initial face.

Then we calculate the Euler characteristic of X' and substitute as follows:

$$\begin{aligned}
\chi(X'_G) &= \sum_{p=0}^{n}(-1)^p(\alpha'_p) \\
&= \alpha'_0 - \alpha'_1 + \alpha'_2 \\
&= (\alpha_0 + \alpha_1 + \alpha_2) - (2\alpha_1 + 6\alpha_2) + (6\alpha_2) \\
&= \alpha_0 - \alpha_1 + \alpha_2 \\
&= \sum_{p=0}^{n}(-1)^p(\alpha_p(X)) = \chi(X_G)
\end{aligned}$$

Hence, we have demonstrated that the Euler characteristic is invariant under barycentric subdivision for a geometric simplicial complex X_G. □

Another argument for invariance can be made much more generally for the abstract simplicial complex.

Proposition 2.17. *If X is an n-simplex then the alternating sum χ'_n of the number of simplices in X' with 'top' vertex at the barycentre of X is $(-1)^n$.*

Proof. We write χ'_n inductively as

$$\chi'_n = \binom{n+1}{n+1} + \binom{n+1}{n}(-\chi'_{n-1}) + \binom{n+1}{n-1}(-\chi'_{n-2}) + \cdots + \binom{n+1}{1}(-\chi'_1), \tag{2}$$

where the term $\binom{n+1}{i}(-\chi'_{i-1})$ counts simplices with the 'second top' vertex at the barycentre of a face of X with the chosen i vertices. For $n = 0$ we obtain $\chi'_0 = \chi_0 = 1$. Then considering the above formula, we have

$$\begin{aligned}
\chi'_1 &= \binom{1+1}{1+1} + \binom{1+1}{1}(-\chi'_{1-1}) \\
&= \binom{2}{2} - \binom{2}{1}(\chi'_0) \\
&= 1 - 2\chi'_0 \\
&= -1.
\end{aligned}$$

By further induction, we deduce $\chi'_q = (-1)^q$ for each q from 1 to $n-1$. Therefore we can rewrite Equation (2) as such

$$\chi'_n = \binom{n+1}{n+1} - \binom{n+1}{n}(-1)^{n+1} - \ldots - \binom{n+1}{1}(-1)^2, \tag{3}$$

(note: $(-1)^{n+1} = (-1)^{n-1}$, so either choice is valid) and condensing all but the first term under a summation, we obtain

$$\chi'_n = \binom{n+1}{n+1} - \sum_{k=1}^{n} (-1)^{k+1} \binom{n+1}{k}.$$

From here, we first shifting the index to $k = 0$, then use Pascal's Rule $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, to unpack the summation and perform the following computation:

$$\begin{aligned}
\chi'_n &= \binom{n+1}{n+1} - \sum_{k=1}^{n} (-1)^{k+1} \binom{n+1}{k} && (4)\\
&= \underbrace{\binom{n+1}{n+1}}_{=1} - \underbrace{-\left[(-1)^{0+1}\binom{n+1}{0}\right]}_{=+1} - \sum_{k=0}^{n} (-1)^{k+1} \binom{n+1}{k} && (5)\\
&= 1 - 1 - \sum_{k=0}^{n} (-1)^{k+1} \left[\binom{n}{k} + \binom{n}{k-1}\right] && (6)\\
&= \underbrace{\sum_{k=0}^{n} (-1)^{k} \binom{n}{k}}_{=(1-1)^n=0 \text{ as } n>0} - \sum_{k=0}^{n} \underbrace{(-1)^{k-1}}_{(-1)^{k+1}=(-1)^{k-1}} \binom{n}{k-1} && (7)\\
&= -\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k-1} && (8)\\
&= -\underbrace{(-1)^{-1}\binom{n}{-1}}_{=0} - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k-1}. && (9)
\end{aligned}$$

Translating the index once more: $k - 1 = a$, so that

$$\begin{aligned}
\chi'_n &= -\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k-1} && (10)\\
&= -\sum_{a=0}^{n-1} (-1)^{a} \binom{n}{a} && (11)\\
&= -\underbrace{\sum_{a=0}^{n} (-1)^{a} \binom{n}{a}}_{=(1-1)^n=0 \text{ as } n>0} + (-1)^n \underbrace{\binom{n}{n}}_{=1} && (12)\\
&= (-1)^n. && (13)
\end{aligned}$$

Hence we have shown $\chi'_n = (-1)^n$ for the top face as desired. $\square$

Corollary 2.18. *If $X_A = (V, S)$ is an abstract simplicial complex then $\chi(X_A) = \chi(X'_A)$.*

Proof. We compute directly as follows

$$\begin{aligned}
\chi(X_A) &= \sum_{\sigma \in S} (-1)^{|\sigma|-1} \\
&= \sum_{\sigma \in S} \sum_{\substack{\tau \in \sigma' \\ (\hat{\sigma} \text{ top vertex})}} (-1)^{|\tau|-1} && \text{by Proposition 2.17} \\
&= \sum_{\tau \in S'} (-1)^{|\tau|-1} \\
&= \chi(X'_A)\,, && \text{where } X'_A = (V', S').
\end{aligned}$$

$\square$

In Section 2.1.1 we touched on isomorphisms between geometric and abstract simplicial complexes. From the constructions, $\chi(X_G) = \chi(S_{X_G})$ and $\chi(X_A) = \chi(|X_A|)$, we have the following corollaries:

Corollary 2.19.

$$\overbrace{\chi(X_G) = \underbrace{\chi(S_{X_G})}_{}}^{\substack{\textit{by Definition 2.11} \\ \textit{and Section 2.1.1}}} = \underbrace{\overbrace{\chi\big(S'_{X_G}\big) = \underbrace{\chi\big(|S'_{X_G}|\big)}_{}}^{\substack{\textit{by Definition 2.11} \\ \textit{and Section 2.1.1}}}}_{\textit{by Corollary 2.18}} = \underbrace{\chi(X'_G)}_{\textit{by Lemma 2.15}}\,.$$

$\square$

Corollary 2.20. *The Euler characteristic of a finite simplicial complex (both geometric and abstract) is invariant under repeated barycentric subdivision.* $\square$

2.2 Finite Posets

A poset is a set equipped with a binary relation, $<$, that satisfies reflexivity, anti-symmetry and transitivity ([DP02, Definition 1.2]). Provided the poset is finite, then it has a well defined Euler characteristic given by the alternating sum of chains, c_n, of length n, as $\sum_{n \geq 0} (-1)^n c_n \in \mathbb{Z}$ ([Lei08, Example 2.3c]).

We will now consider the poset's classifying space (its nerve) to work out an equivalent definition of its Euler characteristic.

2.2.1 Nerve of a Poset and Euler Characteristics

We construct a poset out of an abstract simplicial complex as follows:

Definition 2.21 (Poset (of X_A))**.** Let $X_A = (V, S)$ be an abstract simplicial complex with vertices V and S the non-empty subsets of V. Then the **poset of simplices**, $P_X = S$, is partially ordered by the face relation $\tau \subset \sigma \iff \tau$ is a face of σ for simplices $\sigma, \tau \in S$.

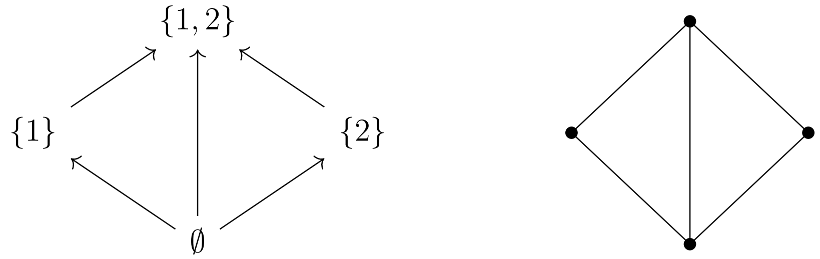
The nerve of a finite poset is then constructed by preserving the structure of n-chains in the poset as n-simplices as follows:

Definition 2.22 (Non-degenerate nerve (Poset))**.** The **non-degenerate nerve**, NP, of a finite poset P is an abstract simplicial complex whose vertices are the elements of P and where $\{p_0, \dots, p_n\}$ is an n-simplex in NP whenever

$$p_0 < p_1 < \cdots < p_n,$$

is a non-degenerate n-chain in P.

Example(s) 2.23. An example poset P on the left, with its non-degenerate nerve NP on the right.



The nerve NP is a simplicial complex with four 0-simplices, five 1-simplices and two 2-simplices. Equated to the four 0-chains, five 1-chains and two 2-chains of P.

If P is a finite poset then P_{NP} is the poset of non-degenerate chains in P, i.e., the elements are non-degenerate chains $p_0 < \cdots < p_n$ and they are partially ordered by the sub-chain relation. There is a map $P_{NP} \to P$ which maps non-degenerate n-chains to their top element: $p_0 < \cdots < p_n \mapsto p_n$ (this is not an isomorphism in general).

If we suppose X is an abstract simplicial complex with corresponding simplicial complex NP_X as in Definition 2.22, then the n-simplices of NP_X are the non-degenerate n-chains $\sigma_0 < \sigma_1 < \cdots < \sigma_n$ in P_X, viz. they are the non-degenerate chains of simplices in X, which are exactly the n-simplices in the barycentric subdivision of X (Definition 2.13). Hence, $NP_X = X'$, and there exists a homeomorphism $|X| \cong |X'| = |NP_X|$.

Consequently, from any abstract simplicial complex X we can produce a poset of simplices P_X and realise its nerve as the barycentric subdivision $NP_X = X'$. Likewise, for any geometric simplicial complex X we have:

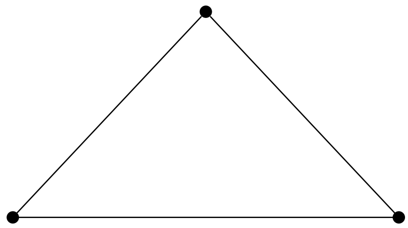
$$|NP_{S_X}| \cong |X'|.$$

We conclude

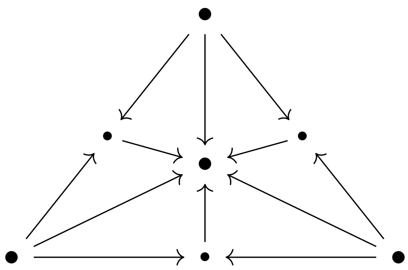
$$\chi(P_X) = \chi(NP_X) = \chi(X') = \chi(X).$$

Example(s) 2.24. For a finite simplicial complex X (geometric or abstract) with a single 2-simplex face, we demonstrate the process $X \to P_X \to NP_X$.

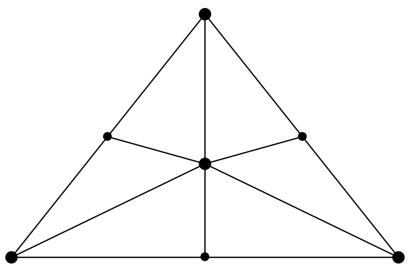
X:



P_X:



NP_X:



2.3 Finite Categories

We introduce finite categories as follows:

Definition 2.25 (Finite Category)**.** A **finite category**, $\mathbb{A}$, consists of the following:

1. A finite collection of objects, $\mathrm{Ob}(\mathbb{A})$.

2. A finite collection of morphisms, $\mathbb{A}(a, b)$, for each $a, b \in \mathrm{Ob}(\mathbb{A})$ that satisfy a composition law: $\mathbb{A}(a, b) \times \mathbb{A}(b, c) \to \mathbb{A}(a, c)$, $\forall a, b, c \in \mathrm{Ob}(\mathbb{A})$, which is associative, namely, given $\theta\colon a \to b$, $\varphi\colon b \to c$ and $\psi\colon c \to d$ we may compose to obtain $\psi \circ (\varphi \circ \theta) = (\psi \circ \varphi) \circ \theta$.

3. An identity morphism, $1_a \in \mathbb{A}(a, a)$ such that $1_a \circ \theta = \theta, \forall \theta \in \mathbb{A}(b, a)$ and similarly $\theta \circ 1_a = \theta, \forall \theta \in \mathbb{A}(a, b)$.

For a finite category, one method of finding its Euler characteristic was presented by Tom Leinster in [Lei08]. My preliminary dissertation, [Edw22], acted as a more readily accessible paper; attempting to guide the reader through this paper with little prior knowledge of categories. The remainder of this subsection presents some material lifted from my preliminary dissertation, with minor editing and comments inserted to catch the reader up when necessary. Example 2.29 is new.

Leinster's initial method involves ordering the objects of the category, $\mathbb{A}$, and taking a matrix M with elements e_{ij} corresponding to the total count of morphisms between the i-th and j-th object (this is $|\mathbb{A}(a, b)| = \zeta(a, b)$). We shall refer to this matrix as the 'matrix of morphisms' for brevity here on out. From this, we compute a sum of 'weightings' and 'coweightings', and provided both sums agree, we take their sum to be the category's Euler characteristic.

Definition 2.26 (Weightings)**.** Let $\mathbb{A}$ be a finite category. A **weighting** on $\mathbb{A}$ is a function $k^\bullet\colon \mathrm{Ob}(\mathbb{A}) \to \mathbb{Q}$, such that $\forall a \in \mathbb{A}$,

$$\sum_b \zeta(a, b) k^b = 1,$$

where $\zeta(a, b) = |\mathbb{A}(a, b)|$ is the count of morphisms from a to b, and $k^\bullet$ denotes the '**weight**' of each object $\bullet \in \mathrm{Ob}(\mathbb{A})$. A **coweighting** (denoted $k_\bullet$) is just the weighting on $\mathbb{A}^{\mathrm{op}}$ (the category $\mathbb{A}$ with the direction of each morphism reversed).

Lemma 2.27. *Let $\mathbb{A}$ be a finite category with weightings and coweightings $k^\bullet$ and $k_\bullet$ respectively. Then $\sum_a k^a = \sum_a k_a$.*

Proof.

$$\sum_b k^b = \sum_b \left(\sum_a k_a \zeta(a,b) \right) k^b = \sum_a k_a \left(\sum_b \zeta(a,b) k^b \right) = \sum_a k_a. \qquad \square$$

Definition 2.28 (Euler Characteristic (Finite Category))**.** A **finite category, $\mathbb{A}$, has Euler characteristic** if it admits both a weighting and coweighting. Its Euler characteristic is then

$$\sum_a k^a = \sum_a k_a = \chi(\mathbb{A}) \in \mathbb{Q},$$

for any weighting $k^\bullet$ and coweighting $k_\bullet$.

Example(s) 2.29. Let $\mathbb{A}$ be a 'pushout' consisting of objects a, b, c with two unique morphisms, $\varphi\colon a \to b$ and $\psi\colon a \to c$, as well as identity morphisms $\mathbb{A}(\bullet, \bullet)$ for each $\bullet \in \{a, b, c\}$. We write its matrix of morphisms M of elements e_{ij} of $\mathbb{A}$ as

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then solve for each weight, obtaining: $M \begin{pmatrix} k^\bullet \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$, as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^a \\ k^b \\ k^c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We find $k^a = -1$, $k^b = 1$ and $k^c = 1$. Hence $\sum_{\bullet \in \{a,b,c\}} k^\bullet = 1$. One can check that the sum of coweightings is also $\sum_\bullet k_\bullet = 1$ by verifying $M^T \begin{pmatrix} k_\bullet \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$. Hence $\chi(\mathbb{A}) = 1$.

Having given the previous presentation of Euler characteristic, we spend the rest of the section detailing the background theory to an alternate Euler characteristic for finite categories which we refer to as the 'Series Euler Characteristic' (later denoted by χ_Σ, Definition 2.40). Defined in Tom Leinster and Clemens Berger's paper [BL08]. It makes use of formal power series, delta sets and the nerves of categories and so we introduce each in sequence.

2.3.1 Formal Power Series

We use [Car63] as a reference for the discussion of formal power series here.

Let K be a field. The formal polynomials in one variable t with coefficients from K behave well under addition and multiplication by scalars, and hence make up a vector space over K with infinite basis. We refer to the set of all such polynomials as $K[t]$. Each polynomial is a finite linear combination of powers of t with finitely many non-zero coefficients $a \in K$, which we write as $\sum_{n\geq 0} a_n t^n$.

Considering multiplication of polynomials, we let $t^p \cdot t^q = t^{p+q}$ and define multiplication in $K[t]$ as the product

$$\left(\sum_p a_p t^p\right) \cdot \left(\sum_q b_q t^q\right) = \sum_n c_n t^n, \qquad \text{where } c_n = \sum_{p+q=n} a_p b_q. \tag{14}$$

For all polynomials $P, Q, R, S \in K[t]$ and all scalars λ we have:

$$\begin{cases} (P+Q) \cdot R = P \cdot R + Q \cdot R, \\ (\lambda S) \cdot R = \lambda(SR), \end{cases}$$

so multiplication of polynomials is commutative and associative. Hence, $K[t]$ forms a commutative algebra with a unit element (let $a_0 = 1$ and $a_{n>0} = 0$) over the field K.

Now we look to formal power series.

Definition 2.30 (Formal Power Series)**.** A **formal power series** over K is an expression

$$a_0 + a_1 t + \cdots = \sum_{n\geq 0} a_n t^n, \quad \forall a_n \in K,$$

for a free variable t where we now no longer require finitely many non-zero coefficients a_n.

Again, we form a vector space over K by defining the sum of two formal power series as: $\left(\sum_{n\geq 0} a_n t^n\right) + \left(\sum_{n\geq 0} b_n t^n\right) = \sum_{n\geq 0} c_n t^n$ where $c_n = a_b + b_n$, and the product of a formal power series by a scalar is to be: $\lambda(\sum_{n\geq 0} a_n t^n) = \sum_{n\geq 0} (\lambda a_n) t^n$. The product of two formal power series is defined as in Equation (14) and we retain that the multiplication is commutative, associative and bilinear with respect to the vector space formed above over K. Thus, as before, we form an algebra with unit element over the field K which we write as $K[\![t]\!]$.

Remark 2.31 ([Car63, pp.10]). "The algebra $K[t]$ is identified with a subalgebra of $K[\![t]\!]$, the subalgebra of formal series whose coefficients are all zero except for a finite number of them."

In certain cases we are able to invert a formal power series (P is invertible in $K[\![t]\!] \iff \exists Q \in K[\![t]\!] \colon P(t) \cdot Q(t) = 1$, the unit element over K). Consider the following examples:

Example(s) 2.32. Let $P(t) \cdot Q(t) = 1$ for polynomials $P, Q \in K[\![t]\!]$.

1. Let $P(t) = (1 - t)$ and $Q(t) = \sum_{n \geq 0} t^n$. Then one can easily verify $P(t) \cdot Q(t) = 1$, hence the series $1 - t$ has an inverse Q in $K[\![t]\!]$.

2. Let $P(t) = \sum_{n \geq 0} a_n t^n$ and $Q(t) = \sum_{n \geq 0} b_n t^n$. Then by comparing coefficients we have

$$\begin{aligned} a_0 b_0 = 1 &\implies a_0 \neq 0, b_0 = \frac{1}{a_0}, \\ a_0 b_1 + a_1 b_0 = 0 &\implies b_1 = -\frac{a_1 b_0}{a_0}, \\ a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 &\implies b_2 = -\frac{a_1 b_1 + a_2 b_0}{a_0}, \\ \vdots \qquad & \quad \vdots \\ a_0 b_i + \cdots + a_i b_0 = 0 &\implies b_i = -\frac{a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_i b_0}{a_0}, \end{aligned}$$

   and so on. Hence P is inverted by Q provided $a_0 \neq 0$.

Lemma 2.33. *The polynomial $P(t) = \sum_{n \geq 0} a_n t^n \in K[\![t]\!]$ is invertible if and only if $a_0 \neq 0$.*

Proof. (See [Car63, pp.14]).

$\implies$ If P is invertible, then there exists some polynomial $Q(t) = \sum_{n \geq 0} b_n t^n$ such that $P(t) \cdot Q(t) = 1$ which implies $a_0 b_0 = 1$ by Equation (14) and so $a_0 \neq 0$ by necessity. $\square$

$\impliedby$ If $a_0 \neq 0$ then we can multiply P by the scalar $(a_0)^{-1}$ to obtain $(a_0)^{-1} P(t) = P_1(t)$. If each coefficient $a_n = 1$ for $n \geq 1$ then P_1 has inverse $1 - t$ as before (Example 2.32-1). Else, there exists some alternate polynomial $U(t)$ with minimal coefficient $b_n \neq 0$ for $n \geq 1$ (has order $\omega(U) \geq 1$) for whom we can define each coefficient b_i for $i \geq 1$ such that $b_i := (a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_i b_0)$. By taking the product $(P(t))(1 - U(t))$

we obtain each $a_i b_i = 0 \forall i = 1, 2, \ldots$ and $a_0 b_0 = 1$ in the same fashion as Example 2.32-2. Hence the polynomial $Q_1(t) = 1 - U(t)$ inverts P_1 and P is invertible by $Q(t) = a_0(1 - U(t))$. □

We now look to define when such polynomials are considered 'rational'.

Definition 2.34 (Rational Formal Power Series)**.** A **formal power series,** $R(t) = \sum_{n \geq 0} a_n t^n \in K[\![t]\!]$ **is rational**, if it has the form

$$R(t) = \frac{P(t)}{Q(t)} = P(t) \cdot \frac{1}{Q(t)},$$

for polynomials $P, Q \in K[t]$ where $Q(t)$ is invertible.

Then, as Leinster writes in [BL08, pp.44]; for any field K, there is a commutative diagram

$$\begin{tikzcd} K[t] \arrow[r, hook] \arrow[d, hook] & K[\![t]\!] \arrow[d, hook] \\ K(t) \arrow[r, hook] & K(\!(t)\!) \end{tikzcd}$$

with $K[t]$ the ring of polynomials over K, and $K[\![t]\!]$ the ring of formal power series as above. We have then $K(t)$ and $K(\!(t)\!)$ to be their respective field of fractions. That is, the field $K(t)$ of rational expressions over K and the field $K(\!(t)\!)$ of rational formal power series (these are the formal Laurent series over K, i.e. finite expressions $\sum_{n \in \mathbb{Z}} a_n t^n$ with finitely many non-zero coefficients a_n for $n \leq 0$).

We will now justify why the field of fractions of $K[\![t]\!]$ is the formal Laurent series over K. First, each Laurent series, $L(t)$, can be written as the quotient of power series,

$$L(t) = \sum_{k=-n}^{\infty} a_k t^k = t^{-n} \cdot \sum_{k=-n}^{\infty} a_k t^{k+n},$$

for which we may write up-to a relabelling of indices, $q = k + n$, that $L(t) = (t^n)^{-1} \cdot \sum_{q \geq 0} a_{q-n} t^q$.

In the other direction, if the quotient of formal power series $\frac{P(t)}{Q(t)} \in K(\!(t)\!)$ then as $Q(t) \neq 0$ we write $Q(t) = t^n \sum_{k>n} a_k t^k$ for some $n \geq 0$ where $a_0 \neq 0$. Moreover, $\sum_{k>n} a_k t^k$ is invertible in $K[\![t]\!]$ because $a_0 \neq 0$ (Lemma 2.33). Then we have another polynomial, $\left(\sum_{k>n} a_k t^k\right)^{-1} = R(t)$. Hence,

$$\frac{P(t)}{Q(t)} = \frac{P(t)}{t^n \sum_{k>n} a_k t^k} = \frac{P(t)R(t)}{t^n},$$

which is a formal Laurent series as above.

2.3.2 Delta Sets

In this section we make use of 'Delta sets' which are generalisations of abstract simplicial complexes in which we allow faces of simplices to be identified. I thank §11 of [Ran92] for its lucid exposition of Delta sets which aided in the writing of this section.

Definition 2.35 (Delta Sets)**.** A **Δ-set**, X, is a collection of sets $\{X_n\}_{n\geq 0}$ together with face maps

$$X_n \;\begin{matrix} \xrightarrow{\partial_0} \\ \vdots \\ \xrightarrow[\partial_n]{} \end{matrix}\; X_{n-1},$$

such that:

$$\partial_i\partial_j = \partial_{j-1}\partial_i, \quad \text{for } i < j, \tag{15}$$

for which we write $\partial_i x$ to mean the ith face of some $x \in X_n$.

We now look at two examples of Δ-sets.

Example(s) 2.36.

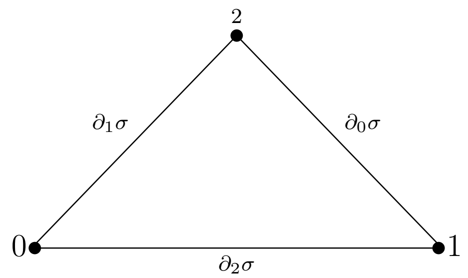
1. X_n is the set of n-simplices in a geometric simplicial complex X_G. To define the maps $\delta_i \colon X_n \to X_{n-1}$ we have to order the vertices in X_G. Then if $\sigma = \langle v_0, \ldots, v_n\rangle \in X_n$ where $v_0 < \cdots < v_n$ in the ordering we write:
$$\delta_i\sigma = \langle v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\rangle \ \in X_{n-1}.$$

2. X_n is the set of n-simplices in an abstract simplicial complex $X_A = (V, S)$. To define the maps $\delta_i \colon X_n \to X_{n-1}$ we order the vertices in V. Then if $\sigma = \{v_0, \ldots, v_n\} \in X_n$ where $v_0 < \cdots < v_n$ in the ordering we write:
$$\delta_i\sigma = \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\} \ \in X_{n-1}.$$

In both cases, we label the vertices of each n-simplex $x \in X_n$, and then define faces $\partial_i x$ to be the face opposite the vertex i, i.e. the face whose vertices are labelled $0, 1, \ldots, i-1, i+1, \ldots n$ (or for a geometric complex, these are the points labelled $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ as in Example 2.36-(1) above).

Remark 2.37. In these examples, $\partial_i x \neq \partial_j x$ when $i \neq j$. So Δ-sets are more general than either geometric or abstract simplicial complexes.

For a basic 2-simplex σ we have the following:



where we take X_0, X_1, X_2 to be the sets of 0,1,2-simplices respectively and label the vertices 0, 1, 2, anticlockwise, with $\partial_i x$ as faces opposite. If we write the 2-simplex as the set $[012]$ and apply ∂_2 to it (implying we are finding the face opposite to the top labelled vertex) we can compute: $\partial_2[012] = [01]$, which is the opposite edge from the 0 to 1 labelled vertices as expected. Applying a second face map, say, ∂_1, we would see $\partial_1\partial_2[012] = \partial_1[01] = [0]$ (and similarly by relation (15) of Definition 2.35 we find $\partial_1\partial_2 = \partial_1\partial_1$ and computing for ourselves we see $\partial_1\partial_1[012] = \partial_1[02] = [0]$, as expected).

Delta sets are also known as 'semi-simplicial' sets [nLa22a]. Here, as in the Examples 2.36, we can construct examples as sets derived from simplicial complexes and see them as sets of $(n+1)$-tuples corresponding to the n-simplices. Another valid definition of the Delta set arises categorically as the presheaf of a specific wide subcategory of the 'simplex category' (which in total is denoted by boldface $\mathbf{\Delta}$) [nLa22b]. The simplex category contains more information than we need, in fact, considering a presheaf, $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, will give us a 'simplicial set' (as opposed to semi-simplicial), which contains additional 'degeneracy' maps; which identify those $(n+1)$-simplices which are degenerate in that they contain repeated vertices. As we do not need this additional structure, we cut it out by defining the desired wide subcategory to be the 'Semi-simplex' category, denoting it as $\mathbf{\Delta}_+$, where the subscript, $+$, depicts our sole interest in the injective maps.

Definition 2.38 (Semi-simplex Categories)**.** Let $\mathbf{\Delta}_+$ be the **Semi-simplex category** whose objects are the finitely ordered sets $[n] = \{0, 1, \dots, n\}$ and whose morphisms are increasing injections generated by basic maps $\partial^i\colon [n-1] \to [n]$ such that for each $m \in [n-1]$ we have:

$$\partial^i(m) = \begin{cases} m & m < i, \\ m+1 & m \geq i, \end{cases}$$

such that $\partial^i\partial^j = \partial^{j+1}\partial^i$ when $i \leq j$.

Remark 2.39. In the (semi)-simplex category we take: $\partial^i\partial^j = \partial^{j+1}\partial^i$, in order to keep the map: $\partial_i\partial_j = \partial_{j-1}\partial_i$, as shown in Definition 2.35 in the (semi)-simplicial set consistent after taking its dual.

Now, should we consider the presheaf $X : \mathbf{\Delta}_+^{\mathrm{op}} \to \mathbf{Set}$ we obtain a categorical definition of a semi-simplicial set (equally that of a Delta set, $\{X_n\}_{n\geq 0}$, where the sets $X_n = X\left([n]\right)$), which can be shown to agree with our earlier definition (See Definitions 2.6 and 2.10 of [Fri08] for such a clear illustration).

2.3.3 Nerves of Finite Categories and Leinster's Theorem of Series Euler Characteristic

Earlier we discussed the non-degenerate nerve of a finite poset (Definition 2.22) as being a type of abstract simplicial complex, here we shall introduce the (non-degenerate) nerve of a finite category as being a (semi) simplicial set. The purpose of this is to, yet again, consider a way in which we can represent a given structure as a type of 'cell complex', which has well understood Euler characteristics (for example, Definition 2.11).

Given a cell complex, we could consider the 'number of cells, c_n, of dimension n' to have some corresponding formal power series $f(t) = \sum_{n\geq 0} c_n t^n$. As is the case for a geometric simplicial complex, X_G, suppose the number of cells are the number of n-simplices, α_n, then we consider the evaluation of the formal power series $f(t)$ at the point $t = -1$ to be $f_{X_G}(-1) = \sum_{n\geq 0}(-1)^n\alpha_n$. Hence, as in Definition 2.10, this formal power series at $t = -1$ is the Euler characteristic of the complex X_G. This is ideal for a finite simplicial complex, as there are only ever finitely many terms and so the sum always converges (in $\mathbb{Z}$).

Clearly, we have some interest in finding when the formal power series $f(t) = \sum_{n\geq 0} c_n t^n$ exists at $t = -1$ or at least when it has analytic continuation to that point. Whenever there are finitely many cells, we have that the formal power series is a polynomial and clearly then exists at $t = -1$. If we continue by considering the Delta set, X, with finite sets, X_n, for all n, then we will define $f_\Delta(t) = \sum_{n\geq 0} |X_n| t^n$. Once more, we note that a finite category, $\mathbb{A}$, has a non-degenerate nerve consisting of finitely many non-degenerate simplices of each dimension (in fact, corresponding to a delta set). We define the power series $f_{\mathbb{A}} := f_{N\mathbb{A}}$.

Let us open by capturing this as a definition, we shall call the Euler characteristic brought about by the formal power series the 'Series Euler Characteristic', denoted χ_Σ and define it as follows:

Definition 2.40 (Series Euler Characteristic)**.** A finite category $\mathbb{A}$ has **series Euler Characteristic** if $f_{\mathbb{A}}(-1) \in \mathbb{Q}$. In that case, its **series Euler**

Characteristic is $\chi_\Sigma(\mathbb{A}) = f_{\mathbb{A}}(-1)$.

The remainder of the section is spent justifying this definition and producing a more explicit formula for $f_{\mathbb{A}}(t)$. Now we look to define the (semi-simplicial) nerve of a category.

Definition 2.41 (Nerve of a Category)**.** The semi-simplicial **nerve of a category** is a semi-simplicial set in which an n-simplex is a chain

$$x_0 \xrightarrow{\varphi_1} x_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} x_n$$

of morphisms in $\mathbb{A}$ for each $x \in X_n$; such an n-simplex is degenerate if and only if some φ_i is an identity. [BL08, pp.45]

The boundary $\partial_i(\varphi_1, \dots, \varphi_n) \coloneqq (\varphi_1, \dots, \varphi_{i+1} \circ \varphi_i, \dots, \varphi_n)$. (Note that this may be degenerate even when $(\varphi_1, \dots, \varphi_n)$ is not, so we cannot restrict to a semi-simplicial set only containing non-degenerate simplices.)

We justify the relation of Euler characteristic between the nerve of a category and the category itself as follows:

Proposition 2.42. *Let $\mathbb{A}$ be a finite and skeletal category containing no endomorphisms except identities. Then $\chi(\mathbb{A}) = \chi(B\mathbb{A}) = \chi(N\mathbb{A})$.*

Proof. Here, $B\mathbb{A}$ is the classifying space of the category $\mathbb{A}$. The classifying space is a geometric realisation of the nerve. When $\mathbb{A}$ is finite, skeletal and all endomorphisms are identities, there are only finitely many non-degenerate simplices in $N\mathbb{A}$, so $B\mathbb{A} = |N\mathbb{A}|$ is a finite simplicial complex and $\chi(B\mathbb{A})$ is well defined. Hence, by our earlier relations of Euler characteristic between complexes, we write $\chi(N\mathbb{A}) = \chi(|N\mathbb{A}|) = \chi(B\mathbb{A})$.

Then the series Euler characteristic $\chi_\Sigma(\mathbb{A}) = \chi_\Sigma(N\mathbb{A}) = \chi_\Sigma(B\mathbb{A})$ because $B\mathbb{A} = |N\mathbb{A}|$. Moreover, by Proposition 2.11 of [Lei08] we know $\chi(B\mathbb{A}) = \chi(\mathbb{A})$. $\square$

The delta set is the corresponding cell complex of finitely many non-degenerate n-simplices of each dimension n with formal power series corresponding to the nerve of the finite category, hence:

$$f_{\mathbb{A}}(t) = f_{N\mathbb{A}}(t) = \sum_{n \geq 0} c_n t^n \in K[\![t]\!].$$

We now take a detour to discuss some relevant facts about matrices. Let $n \in \mathbb{N}$ and let K be a commutative ring with $M \in \mathrm{Mat}_n(K)$ an $n \times n$ matrix with elements $M_{ij} \in K$ for all i, j from 1 to n. We write $s\colon \mathrm{Mat}_n(K) \to K$

for the K-linear map defined by $s(M) = \sum_{i,j} M_{ij}$ (hence s takes the sum over all elements in the matrix). Then all matrices $M \in \mathrm{Mat}_n(K)$ have an adjugate, $\mathrm{adj}(M) \in \mathrm{Mat}_n(K)$ defined by:

$$\mathrm{adj}(M) = \left((-1)^{i+j} \cdot \det(D_{ji})\right)_{1 \le i,j \le n},$$

where D_{ji} is the $(n-1) \times (n-1)$ matrix of M produced by removing the jth row and ith column. Consequently, the adjugate matrix satisfies the following equation:

$$M \cdot \mathrm{adj}(M) = \mathrm{adj}(M) \cdot M = \det(M) \cdot I_n, \tag{16}$$

for an $n \times n$ identity matrix I_n.

Lemma 2.43. *Let $M \in Mat_n(K)$ be a square matrix over a field K. Then the sum $\sum_{n\in\mathbb{N}} s(M^n)t^n \in K[\![t]\!]$ is rational (Definition 2.34).*

We thank Tom Leinster for this proof as [BL08, Proof of Lemma 2.1]:

Proof. Write $F(t) = \sum_{n\in\mathbb{N}} M^n t^n \in \mathrm{Mat}_n(K[\![t]\!])$. Then $(I_n - Mt)F(t) = I_n$, so $\det(I_n - Mt) \cdot F(t) = \mathrm{adj}(I_n - Mt)$.

Applying the K-linear map $s\colon \mathrm{Mat}_n(K[\![t]\!]) \to K[\![t]\!]$, we obtain

$$\det(I_n - Mt) \cdot s(F(t)) = s(\mathrm{adj}(I_n - Mt)).$$

But $s(F(t)) = \sum_{n\in\mathbb{N}} s(M^n)t^n$ and $\det(I_n - Mt)$ is not the zero polynomial (since its value at $t = 0$ is 1), so $\sum_{n\in\mathbb{N}} s(M^n)t^n$ is rational and equal to

$$\frac{s(\mathrm{adj}(I_n - Mt))}{\det(I_n - Mt)} \in K(t). \qquad \square$$

Returning from this matrix detour, we now propose Leinster's primary theorem:

Theorem 2.44. *For any finite category $\mathbb{A}$, the formal power series $f_{\mathbb{A}}$ is rational (over $\mathbb{Q}$).*

Proof. As before (with (co)weightings, Definition 2.26), we order the objects of $\mathbb{A}$ and take a matrix of morphisms, M containing elements $e_{ij} \in K$ (in fact, in $\mathbb{Z}$ by construction) the total count of morphisms between the i-th and j-th objects ($|\mathbb{A}(a,b)| \in \mathbb{Z}$). If we remove the identity morphisms, we obtain a matrix $(M - I)$ with $(M - I)_{ij}$ many morphisms from e_i to e_j. By the chain relation of Definition 2.41 (non-degenerate n-simplices form chains of non-identity morphisms in $\mathbb{A}$) we conclude that the number of non-degenerate n-simplices between e_i and e_j is therefore $((M - I)^n)_{ij}$. Hence the total number, c_n, of non-degenerate n-simplices is their sum, $s((M - I)^n)$. By Leinster's result in Lemma 2.43 we have demonstrated $f_{\mathbb{A}}$ is rational. $\square$

As stated at the beginning of this subsection, a formal power series, $f_{\mathbb{A}}(t) = \sum_{n\geq 0} c_n t^n \in \mathbb{Q}[\![t]\!]$, does not converge in general, we need additional criteria. Since by the above, $f_{\mathbb{A}}(t) = \sum_{n\geq 0} c_n t^n \in \mathbb{Q}[\![]\!]$, it makes sense to define $\chi_\Sigma(\mathbb{A}) = f_{\mathbb{A}}(-1) \in \mathbb{Q}$ provided after simplifying the RHS of equation (17) the denominator does not vanish at $t = -1$. We consider that a category with non-degenerate nerve corresponding to a Delta set will converge at $t = -1$ as required. By combining the proofs above, we may write $f_{\mathbb{A}}$ as follows: Consider the matrix of morphisms of $\mathbb{A}$, then we write

$$f_{\mathbb{A}}(t) = \frac{s(\mathrm{adj}(I_n - (M - I_n)t)}{\det(I_n - (M - I_n)t)}, \tag{17}$$

provided $f_{\mathbb{A}}(-1) \in \mathbb{Q}$, and we define $f_{\mathbb{A}}(-1) = \chi_\Sigma(\mathbb{A})$ by Definition 2.40.

2.3.4 Comparing Euler Characteristics and Series Euler Characteristics

Let us consider a number of examples of finite categories for which we can attempt to compute a sensible (series) Euler characteristic.

One can consider the finite group, G, to be a finite category containing one object and $|G|$-many automorphisms, where $|G|$ is the order of the group. Moreover, one can consider its classifying space, BG, to be a simplicial set, wherein non-degenerate n-simplices are n-tuples of non-identity elements of G. Hence we consider its nerve to have $|G| - 1$ many non-degenerate n-simplices. Consider then its Euler characteristic as the alternating sum of n-simplices we find: $\chi(G) = \sum_{n\geq 0}(-1)^n(|G| - 1)^n = |G|^{-1}$, by evaluating it as the sum for a geometric series. Likewise, when we look to compute its series Euler characteristic we write $f_G(t) = \sum_{n\geq 0} c_n t^n = (1 - (|G| - 1)t)^{-1}$, and evaluating at $t = -1$ we find:

$$\chi_\Sigma(G) = f_G(-1) = \frac{1}{|G|}.$$

so that both definitions, in fact, agree.

We shall now explore a handful of example categories now.

Example(s) 2.45.

1. Let $\mathbb{A}_1$ be a 3-object category with corresponding matrix of morphisms and series Euler characteristic:

$$M_{\mathbb{A}_1} = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 12 & 9 \end{pmatrix}, \qquad \chi_\Sigma(\mathbb{A}_1) = \frac{1}{7},$$

However, if we compute its regular Euler characteristic we find $\chi(\mathbb{A}_1) = \frac{1}{6}$. So they both exist but happen to disagree.

2. Let $\mathbb{A}_2$ be a 4-object category without a weighting as in [Lei08, Example 1.11(d)]. Then it has corresponding matrix and series Euler characteristic:
$$M_{\mathbb{A}_2} = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \chi_\Sigma(\mathbb{A}_2) = 1,$$
with no regular $\chi(\mathbb{A}_2)$ as the weighting and coweightings do not agree.

3. Now consider the categorical sum (coproduct) of these two categories: $\mathbb{A}_1 \oplus \mathbb{A}_2$ (this can be considered as their disjoint sum). Where its matrix of morphisms: $M_{\mathbb{A}_1 \oplus \mathbb{A}_2} = \left[\begin{smallmatrix} M_{\mathbb{A}_1} & 0 \\ 0 & M_{\mathbb{A}_2} \end{smallmatrix}\right]$ is a block matrix. Computing its series Euler characteristic we obtain:
$$\chi_\Sigma(\mathbb{A}_1 \oplus \mathbb{A}_2) = \frac{8}{7}.$$
Again, with no such regular Euler characteristic.

4. One can (quite easily) uncover examples where both the series and regular Euler characteristic exist and agree. Let $\mathbb{A}_3$ be a 2-object category with generic elements such that both $a, d \geq 1$ then we can compute their series and regular Euler characteristics:
$$M_{\mathbb{A}_3} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \chi(\mathbb{A}_3) = \frac{(a+d)-(b+c)}{ad-bc} = \chi_\Sigma(\mathbb{A}_3)\,.$$
Provided $ad \neq bc$.

5. For the above category, $\mathbb{A}_3$, let $a = d = 2$ and set $c = 1$ and $b = 4$, so that $ad = bc$. Then clearly we have found a category that has no regular nor series Euler characteristic (nor does it have any weighting or coweighting).

Proposition 2.46. *The series Euler characteristic behaves well with respect to the categorical sum (coproduct) of finite categories. For finite categories, $\mathbb{A}_1$ and $\mathbb{A}_2$ with series Euler characteristic $\chi_\Sigma(\mathbb{A}_1)$ and $\chi_\Sigma(\mathbb{A}_2)$ respectively, we find:*
$$\chi_\Sigma(\mathbb{A}_1 \oplus \mathbb{A}_2) = \chi_\Sigma(\mathbb{A}_1) + \chi_\Sigma(\mathbb{A}_2)\,,$$
provided both corresponding matrices of morphisms are invertible.

Proof. Separately, if both categories have invertible matrix of morphisms then we note that their series Euler character exists trivially as $f_{\mathbb{A}_i}(-1) = \chi_\Sigma(\mathbb{A}_i)$ exists. Write $M_{\mathbb{A}_i}$ for the matrix of morphisms of each category $\mathbb{A}_i$ and write n_i to represent the total number of objects in a given finite category, i.e. $n_i = |\mathrm{Ob}(\mathbb{A}_i)|$.

Consider the block matrix $M_{\mathbb{A}_1 \oplus \mathbb{A}_2} = \left[\begin{smallmatrix} M_{\mathbb{A}_1} & 0 \\ 0 & M_{\mathbb{A}_2} \end{smallmatrix}\right]$ for the $n_i \times n_i$ square matrices corresponding to each $\mathbb{A}_i$ (where n_1 need not be equal to n_2). As such, this block matrix represents the matrix of morphisms of the sum of two finite categories $\mathbb{A}_1$ and $\mathbb{A}_2$.

If each matrix of morphisms, $M_{\mathbb{A}_i}$, is invertible, then the inverse of their block sum, $M^{-1}_{\mathbb{A}_1 \oplus \mathbb{A}_2}$, is the block sum of their inverses, $M^{-1}_{\mathbb{A}_i}$, namely:

$$M^{-1}_{\mathbb{A}_1 \oplus \mathbb{A}_2} = \begin{bmatrix} M^{-1}_{\mathbb{A}_1} & 0 \\ 0 & M^{-1}_{\mathbb{A}_2} \end{bmatrix}.$$

Consider that the adjugate matrix satisfies the equation (16) and that we can decompose the determinant of this block matrix, $M_{\mathbb{A}_1 \oplus \mathbb{A}_2}$, as $\det(M_{\mathbb{A}_1 \oplus \mathbb{A}_2}) = \det(M_{\mathbb{A}_1}) \cdot \det(M_{\mathbb{A}_2})$, we write its adjugate as follows:

$$\begin{aligned}
\mathrm{adj}(M_{\mathbb{A}_1 \oplus \mathbb{A}_2}) &= M^{-1}_{\mathbb{A}_1 \oplus \mathbb{A}_2} \cdot \det(M_{\mathbb{A}_1 \oplus \mathbb{A}_2}) \cdot I_{n_1+n_2} \\
&= \begin{bmatrix} \det(M_{\mathbb{A}_1}) \cdot \det(M_{\mathbb{A}_2}) \cdot M^{-1}_{\mathbb{A}_1} & 0 \\ 0 & \det(M_{\mathbb{A}_1}) \cdot \det(M_{\mathbb{A}_2}) \cdot M^{-1}_{\mathbb{A}_2} \end{bmatrix} \\
&= \begin{bmatrix} \det(M_{\mathbb{A}_2}) \cdot \mathrm{adj}(M_{\mathbb{A}_1}) & 0 \\ 0 & \det(M_{\mathbb{A}_1}) \cdot \mathrm{adj}(M_{\mathbb{A}_2}) \end{bmatrix}.
\end{aligned}$$

Should we apply the K-linear map, $s\colon s(M) = \sum_{i,j} M_{ij}$, to the above we find:

$$s(\mathrm{adj}(M_{\mathbb{A}_1 \oplus \mathbb{A}_2}) = \det(M_{\mathbb{A}_2}) \cdot s(\mathrm{adj}(M_{\mathbb{A}_1})) + \det(M_{\mathbb{A}_1}) \cdot s(\mathrm{adj}(M_{\mathbb{A}_2})),$$

(one may easily verify that this map is distributive across a block matrix). By rearranging our definition of adjoint such that $M^{-1} = \frac{1}{\det(M)}\mathrm{adj}(M)$, then taking its K-linear map, s, once more we write:

$$\begin{aligned}
s(M^{-1}_{\mathbb{A}_1 \oplus \mathbb{A}_2}) &= \frac{s(\mathrm{adj}(M_{\mathbb{A}_1 \oplus \mathbb{A}_2}))}{\det(M_{\mathbb{A}_1 \oplus \mathbb{A}_2})}, \\
&= \frac{\det(M_{\mathbb{A}_2}) \cdot s(\mathrm{adj}(M_{\mathbb{A}_1})) + \det(M_{\mathbb{A}_1}) \cdot s(\mathrm{adj}(M_{\mathbb{A}_2}))}{\det(M_{\mathbb{A}_1}) \cdot \det(M_{\mathbb{A}_2})} \\
&= \frac{s(\mathrm{adj}(M_{\mathbb{A}_1}))}{\det(M_{\mathbb{A}_1})} + \frac{s(\mathrm{adj}(M_{\mathbb{A}_2}))}{\det(M_{\mathbb{A}_2})}.
\end{aligned}$$

Therefore, by an earlier result (Equation 17): write $f_{\mathbb{A}_1 \oplus \mathbb{A}_2}(-1) = \chi_\Sigma(\mathbb{A}_1 \oplus \mathbb{A}_2)$ and $\chi_\Sigma(\mathbb{A}_1 \oplus \mathbb{A}_2) = \chi_\Sigma(\mathbb{A}_1) + \chi_\Sigma(\mathbb{A}_2)$, as desired. □

Corollary 2.47. *The categorical sum of finitely many finite categories with invertible matrix of morphisms has series Euler characteristic equal to the sum of each individual finite category:* $\chi_\Sigma\left(\bigoplus_{i\in I}\mathbb{A}_i\right) = \sum_{i\in I}\chi_\Sigma(\mathbb{A}_i)$.

Proof. Consider a repeated application of Proposition 2.46. □

We shall now attempt to answer the question: is it possible to find additional morphisms that connect the two disjoint categories without impacting their combined series Euler characteristic in general?

One notes that such a block matrix would have additional conformable blocks B and C in $M = \left[\begin{smallmatrix} M_{\mathbb{A}_1} & B \\ C & M_{\mathbb{A}_2}\end{smallmatrix}\right]$; which contains these additional morphisms between the categories. If $M_{\mathbb{A}_1}$ and $M_{\mathbb{A}_2}$ are invertible then

$$M^{-1} = \begin{bmatrix} (M_{\mathbb{A}_1} - BM_{\mathbb{A}_2}^{-1}C)^{-1} & 0 \\ 0 & (M_{\mathbb{A}_2} - CM_{\mathbb{A}_1}^{-1}B)^{-1}\end{bmatrix} \cdot \begin{bmatrix} I & -BM_{\mathbb{A}_2}^{-1} \\ -CM_{\mathbb{A}_1}^{-1} & I\end{bmatrix}$$

when all blocks are defined. If both $BM_{\mathbb{A}_2}^{-1}$ and $CM_{\mathbb{A}_1}^{-1}$ are zero matrices then the block matrix is inverted as before. However, $M_{\mathbb{A}_i}$ is invertible, therefore we may write, $BM_{\mathbb{A}_2}^{-1} = 0$ if and only if $BM_{\mathbb{A}_2}^{-1}M_{\mathbb{A}_2} = 0M_{\mathbb{A}_2}$; so that $B = 0$ and similarly for $C = 0$. Therefore we cannot add any additional morphisms between categories in general (though one might find specific cases where additional morphisms coincidentally have no impact, of course).

What if we instead manufacture a matrix that does not change the series Euler characteristic but allows B and C to contain morphisms? Let us consider an alternative partitioning of the block matrix. Let n be the count of objects in $\mathbb{A}$ and m the count of objects in $\mathbb{B}$. Then provided both have invertible matrices of morphisms, we write the categorical sum $\mathbb{A}\oplus\mathbb{B}$ as the block matrix of the following invertible matrices of morphisms where $a_{i,j} = |\mathbb{A}(i,j)|$ and $b_{i,j} = |\mathbb{B}(i,j)|$:

$$M_{\mathbb{A}} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}\end{pmatrix}, \quad M_{\mathbb{B}} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,m}\end{pmatrix}.$$

Then the block matrix $M_{\mathbb{A}\oplus\mathbb{B}}$ has series Euler characteristic $\chi_\Sigma(\mathbb{A}\oplus\mathbb{B})$ so long as $M_{\mathbb{A}}$ and $M_{\mathbb{B}}$ are invertible. Realise then that the block matrix is 'partitioned' by its initial construction. Once constructed, however, it makes no difference as to how we choose to split up the four conforming blocks (provided nothing changes).

By our initial construction we consider the block 'partitioned' by the $n\times n$ and $m\times m$ blocks, joined across the diagonal, with zero blocks B and C

conforming to the remaining space. Now, let us instead partition the block matrix into sub-blocks of size $(n \pm x) \times (n \pm x)$ and $(m \mp x) \times (m \mp x)$ for some $x \in \mathbb{N}$ less than both m and n. Provided that these new diagonal blocks are still invertible, then their combined series Euler characteristic is the same (as the total block matrix is equivalent). We differentiate an invertible sub-matrix of the matrix of morphisms of a finite category by the use of a dash:

$$M_{\mathbb{A},\mathbb{B}} = \left[\begin{array}{c:c} M_{\mathbb{A}} & 0 \\ \hdashline 0 & M_{\mathbb{B}} \end{array}\right] = \left[\begin{array}{c:c} M_{\mathbb{A}'} & B \\ \hdashline C & M_{\mathbb{B}'} \end{array}\right].$$

For example, consider a positive shift of $x = 1$ where both matrix of morphisms $M_{\mathbb{A}'}$ and $M_{\mathbb{B}'}$ are again invertible. Then one might view the shifted partition as follows:

$$\left(\begin{array}{ccc:ccc} a_{1,1} & \cdots & a_{1,n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & \cdots & 0 \\ \hdashline 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,m} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{m,1} & \cdots & b_{m,m} \end{array}\right) = \left(\begin{array}{cccc:cc} a_{1,1} & \cdots & a_{1,n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,m} \\ \hdashline \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{m,1} & \cdots & b_{m,m} \end{array}\right)$$

Then (for a positive x) we view all conforming B as $(n + x) \times (m - x)$ matrices that contain $(n) \times (m - x)$ zero sub-matrices with x-many additional rows adjoined at the bottom containing the elements from the top x rows of $M_{\mathbb{B}}$. Similarly, all C matrices are $(m - x) \times (n + x)$ matrices that contain $(m - x) \times (n)$ zero sub-matrices with x-many additional columns adjoined to the right containing elements from the leftmost x columns of $M_{\mathbb{B}}$. Likewise, a *negative* shift of $x = 1$ would instead give us blocks B and C containing zero sub-matrices with additional rows and columns containing elements from the final rows and columns of $M_{\mathbb{A}}$.

One could contextualise this partition in two ways. Of course, quietly nothing has changed, we can still consider the block matrix to represent two categories $\mathbb{A}$ and $\mathbb{B}$ with no morphism between them. Alternatively, we can consider the new partition into invertible categories, $\mathbb{A}'$ and $\mathbb{B}'$, as a relabelling of the elements, and we now have a block matrix containing two different categories with morphisms contained in B and C connecting them which has the same series Euler characteristic as the initial block matrix. However, this is not as exciting as one might hope. In the above case where we shifted positively by 1, the re-imagining requires that we isolate an object from $\mathbb{B}$ and place it alongside the objects of $\mathbb{A}$ and then it has morphisms that solely join it to objects in $\mathbb{B}$ anyway. If we are to remove any of those morphisms (perhaps by taking B and C as zero matrices again) then the series Euler characteristic of the block will remain the same if and only if the newly partitioned matrices coincidentally have the same block sum, which is not guaranteed. Though this is no counter example to the earlier claim of B and C being totally empty, we have at least found a method with which to consistently write blocks with non-zero B and C such that the series Euler characteristic does not change.

These results seem sufficient to suggest that the inclusion of additional morphisms between objects from disjoint categories will, in general, alter their combined series Euler characteristic, and (for the time being) any such case otherwise is subject to chance. Further work may illustrate a connection or pattern between the types of categories that share series Euler characteristic but differ only in these additional morphisms. One might approach this from either direction. Though, in the same manner we explored, it may be easier to say something about the classes of matrices that exhibit this property than it is to find a satisfying collection of categories (provided they do not already neatly exist outside of the writer's knowledge!).

3 Wallpapers and Orbifolds

John Conway showed in [CH02] that the 17 wallpaper (or plane crystallographic) groups (and more) can be enumerated by considering the topology of orbifolds. This was a sensible (in retrospect) but unique approach, a more traditional approach to the enumeration can be found in chapter 26 of [Arm88] (with necessary background presented in previous chapters). We spend this section outlining this approach with some examples and then verify that the series Euler characteristic of a category defined from a triangulation of the orbifold is consistent with this work.

The wallpaper groups, W are discrete group of isometries of the Euclidean plane containing two independent translations, t_1, t_2. The subgroup generated by these translations is necessarily normal, so that $\mathbb{R}^2/W \cong (\mathbb{R}^2/\langle t_1, t_2\rangle) \,/\, (W/\langle t_1, t_2\rangle)$, where $G = (W/\langle t_1, t_2\rangle)$ is now a finite group acting on the torus, $T^2 = \mathbb{R}^2/\langle t_1, t_2\rangle$. Conway's method relies on a 'defect' formula, which takes advantage of the Euler characteristic of the underlying manifold and reducing it (to the Euler characteristic of the orbifold) by considering the properties induced by the orbits of the wallpaper groups. We shall now show that the series Euler characteristic work agrees with these results.

In Conway's notation we write $\circ$ to depict a handle, $\times$ to depict a crosscap (for non-orientable surfaces) and $*$ to depict a hole. The numbers before and after a hole are the cone and corner points respectively. For example: $4 * 2$ is the orbifold of the disk with order 4 cone and order 2 corner points and $*\times$ depicts a Möbius strip.

Conway's Defect formula [CH02, pp.252] is as follows:

Definition 3.1 (Defect Formula)**.** Let X be a compact surface with finite group G acting on X. Then Q is the orbifold quotient and we compute the **Orbifold Euler Characteristic** as follows:

$$\chi(Q) = \frac{\chi(X)}{|G|} = \chi(X/G) - \sum \text{defects}^{(\text{cone and corner})}$$
$$= 2 - \sum \text{defects}^{(\text{all})}.$$

Conway points out that one can always begin with a sphere and (hence $\chi(X/G) = 2$) and then one can consider adjoining features to the sphere by the orbifolds as 'defects' too. Though it is equally valid to start with an already 'defected' sphere and consider solely the possible cone and corner points that remain. Given an orbifold of the form $\circ \cdots \circ\, AB \cdots * ab \cdots * \alpha\beta \cdots \times \cdots \times$, we now note the contribution of its defect for each feature as:

$$\begin{cases} \circ & \text{defect} = 2, \\ *, \times & \text{defect} = 1, \\ \text{cone points: } AB\ldots & \text{defect} = \frac{n-1}{n}, \\ \text{corner point: } ab\ldots & \text{defect} = \frac{n-1}{2n}. \end{cases}$$

These defects come from the points of X with isotropy groups.

As the interest is to investigate the wallpaper groups, we take our compact X to be the torus, T^2, so that $\chi(X) = 0$. Moreover, as $\chi(X/G) \geq 0$ there are only a handful of possible surfaces we can consider. With aid of the Classification Theorem (see [GX13, pp.96, Theorem 6.3]) we deduce that there are seven such eligible surfaces. Table 1 organises the seven surfaces by their Euler Characteristic.

$\chi(X/G) = 2$	$\chi(X/G) = 1$	$\chi(X/G) = 0$
Sphere.	Disk, Real Projective Plane.	Torus, Möbius Strip, Klein Bottle, Annulus.

Table 1: Table of unique surfaces with positive Euler characteristic.

Therefore, when we wish to find all orbifold quotients Q with $\chi(Q) = 0$ we can look to solve for all possible $\chi(X/G) = \sum$ defects.

We explore one explicit calculation of the defect formula below.

Example(s) 3.2. The orbifold of the disk $4*2$ has defect as follows

$$\chi(Q) = 2 - \underbrace{\left(\frac{4-1}{4}\right)}_{\substack{\text{order } 4\\ \text{cone point}}} + \underbrace{1}_{\substack{*\\ \text{contribution}}} + \underbrace{\left(\frac{2-1}{2\cdot 2}\right)}_{\substack{\text{order } 2\\ \text{corner point}}} = 0.$$

Which shows that this orbifold arises from a wallpaper group as $\sum$ defects $=$ 2. Equally, for an orbifold of the disk, we could have also viewed it as $\chi(\text{Disk}) = 1 = \sum$ defects for just the cone and corner point defects.

3.1 Teardrop Orbifold

Before exploring other examples, let us consider the teardrop orbifold, that is the quotient space of a 2-sphere with one cone point adjoined of order n. We thank Ieke Moerdijk for Section 3.1 of [MP99] and dedicate our own Section 3.1 to containing the quotient and triangulation of the teardrop orbifold as in Figure 1.

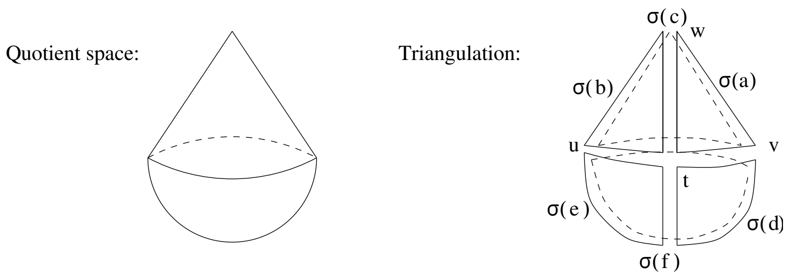


Figure 1: The quotient space of the teardrop orbifold with a triangulation. [MP99, Fig. 2]

Here, we depict the triangulation of the teardrop to consist of the simplices $\sigma(a)$, $\sigma(b)$, $\sigma(d)$ and $\sigma(e)$ as those on the front, and simplices $\sigma(c)$ and $\sigma(f)$ are those on the back. Then each face is a union: $\sigma(i_0) \cap \cdots \cap \sigma(i_n) = \sigma(i_0 \dots i_n)$, and we have $t = \sigma(abde)$, $u = \sigma(bcef)$, $v = \sigma(acdf)$ and $w = \sigma(abc)$.

As Leinster writes in [Lei08, pp.36], we can use the finite triangulation of a compact orbifold to obtain a poset structure. We let P be the poset of simplices (Definition 2.21) of the triangulation of an orbifold Q with P^{op} its dual. From P^{op} we consider the functor $G\colon P^{\text{op}} \longrightarrow \mathbf{FinGp}$ that takes each simplex, σ, in P^{op} to the group of order 1 whenever the simplex does not correspond to a cone point and takes each simplex corresponding to a cone point to a cyclic group of the same order. That is:

$$
\begin{aligned}
G\colon P^{\mathrm{op}} &\longrightarrow \mathbf{FinGp}\\
\sigma &\longmapsto \begin{cases} 1 & \sigma \text{ is not a cone point,}\\ C_n & \sigma \text{ is a cone point of order } n. \end{cases}
\end{aligned}
$$

Then we consider the corresponding category $\mathbb{E}(G)$ to have objects the same as P^{op} (that is, simplices) with morphisms pairs (σ, g) for $\sigma \geq \tau \in P$ and $g \in G(\tau)$.

Consider then, for the morphisms $\sigma \to \tau$ in P^{op} corresponding to $\tau \leq \sigma$ in P (that is, τ is a face of σ) we will have exactly $|I_\tau|$ morphisms from $\sigma \to \tau$ in $\mathbb{E}(G)$ where I_τ are the isotropy groups of the simplex τ.

We now have our recipe to write the corresponding matrix of morphisms of the category of elements, $\mathbb{E}(G)$, as follows:

$$
M_{\mathbb{E}(G)} = \left(\begin{array}{c|c|c|c}
I_3 & \begin{matrix}1&1&1&0&0&0\\0&0&1&1&1&0\\1&0&0&0&1&1\end{matrix} & \begin{matrix}1&1&0\\0&1&1\\1&0&1\end{matrix} & \begin{matrix}n\\n\\n\end{matrix}\\ \hline
0 & I_6 & \begin{matrix}1&1&0\\0&1&1\\1&0&1\end{matrix} & \begin{matrix}0\\0\\0\end{matrix}\\ \cline{3-4}
 & & I_3 & \begin{matrix}n\\n\\n\end{matrix}\\ \hline
0 & 0 & I_3 & \begin{matrix}0\\0\\0\end{matrix}\\ \cline{3-4}
 & & \begin{matrix}0&0&0\end{matrix} & n
\end{array}\right)
$$

Then we find that both its regular and series Euler characteristics agree and are: $\chi_\Sigma\big(M_{\mathbb{E}(G)}\big) = \chi\big(M_{\mathbb{E}(G)}\big) = \frac{1}{n}$, as one would expect.

Remark 3.3. Each cone point of order n corresponds to an isotropy group C_n of order n. It can likewise be shown that each corner point of order n possesses isotropy group C_{2n} of order $2n$.

3.2 Wallpaper Orbifolds

The remainder of this paper presents the full table of all orbifolds with wallpaper groups and then presents seven matrices of morphisms corresponding to wallpaper groups, one for each of the seven surfaces in Table 1.

17 Orbifolds				
$*632$				632
$*442$		$4*2$		442
$*333$		$3*3$		333
$*2222$		$2*2$	$22*$	2222
$**$	$*\times$	$\times\times$	$22\times$	$\circ$

Table 2: Recreation of Table II from [CH02, pp.254]; amended to include the missing $22\times$ orbifold of $\mathbb{R}P^2$.

We now present seven examples. One for each type of possible surface and orbifold to admit an appropriate wallpaper group. They each have a corresponding matrix of morphisms of the category of elements arising from the triangulation of their respective orbifolds. We verify that each has regular and series Euler characteristics of zero which agrees with Conway's defect formula and the general topology of each orbifold. The remaining 10 wallpaper groups are formed in much the same way, in fact, they are just alternate examples of the sphere and disk orbifolds, and, as such, are no more instructive than the seven presented. Hence, they remain as an exercise of understanding for the reader!

Example(s) 3.4.

1. The Torus, $\circ$.

$$M_\circ = \left(\begin{array}{c|c|c} I_3 & \begin{matrix}1&1&0\\0&1&1\\1&0&1\end{matrix} & \begin{matrix}1&1\\1&1\\1&1\end{matrix} \\ \hline 0 & I_3 & \begin{matrix}1&1\\0&0\\1&1\end{matrix} \\ \hline 0 & 0 & I_2 \end{array}\right), \qquad \begin{matrix}\chi(M_\circ)=0,\\ \chi_\Sigma(M_\circ)=0.\end{matrix}$$

2. The Real Projective Plane: $22\times$.

$$M_{22\times} = \left(\begin{array}{c|c|c} 2I_2 & \begin{matrix}1&1&0\\0&1&1\end{matrix} & \begin{matrix}1&1\\1&1\end{matrix} \\ \hline 0 & I_3 & \begin{matrix}1&1\\1&1\\1&1\end{matrix} \\ \hline 0 & 0 & I_2 \end{array}\right), \qquad \begin{matrix}\chi(M_{22\times})=0,\\ \chi_\Sigma(M_{22\times})=0.\end{matrix}$$

3. The Klein Bottle: $\times\times$.

$$M_{\times\times} = \left(\begin{array}{c|ccccc|cc}
 & 4 & 4 & 0 & 0 & 0 & 4 & 0 \\
4I_4 & 4 & 0 & 0 & 4 & 4 & 4 & 4 \\
 & 0 & 0 & 4 & 4 & 0 & 0 & 4 \\
 & 0 & 4 & 4 & 0 & 4 & 4 & 4 \\ \hline
 & & & & & & 2 & 0 \\
 & & & & & & 2 & 0 \\
0 & & & 2I_4 & & & 0 & 2 \\
 & & & & & & 0 & 2 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline
0 & & & 0 & & & \multicolumn{2}{c}{I_2}
\end{array}\right), \qquad \begin{array}{c} \chi_\Sigma(M_{\times\times}) = 0, \\ \chi(M_{\times\times}) = 0. \end{array}$$

4. The Möbius strip: $*\times$.

$$M_{*\times} = \left(\begin{array}{cc|ccc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{I_2}} & 1 & 1 & 1 & 2 \\
 & & 1 & 1 & 1 & 2 \\ \hline
 & & & & & 2 \\
\multicolumn{2}{c|}{0} & & I_3 & & 1 \\
 & & & & & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \qquad \begin{array}{c} \chi_\Sigma(M_{*\times}) = 0, \\ \chi(M_{*\times}) = 0. \end{array}$$

5. The Annulus: $**$.

$$M_{**} = \left(\begin{array}{cc|ccc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$2I_2$}} & 0 & 2 & 2 & 2 \\
 & & 2 & 2 & 0 & 2 \\ \hline
 & & 1 & 0 & 0 & 1 \\
\multicolumn{2}{c|}{0} & 0 & 2 & 0 & 1 \\
 & & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \qquad \begin{array}{c} \chi_\Sigma(M_{**}) = 0, \\ \chi(M_{**}) = 0. \end{array}$$

6. The sphere with four cone points of order 2: 2222.

$$M_{2222} = \left(\begin{array}{c|c|c} I_4 & \begin{matrix}1&1&1&0&0&0\\0&0&1&1&1&0\\1&0&0&0&1&1\\0&1&0&1&0&1\end{matrix} & \begin{matrix}2&2&2&0\\0&2&2&2\\2&0&2&2\\2&2&0&2\end{matrix} \\ \hline 0 & I_6 & \begin{matrix}2&2&0&0\\0&2&2&0\\0&0&2&2\\2&0&2&0\\2&0&0&2\\0&2&0&2\end{matrix} \\ \hline 0 & 0 & 2I_4 \end{array}\right), \qquad \begin{matrix}\chi_\Sigma(M_{2222}) = 0,\\ \chi(M_{2222}) = 0.\end{matrix}$$

7. The disk with 3 corner points: $*333$.

$$M_{*333} = \left(\begin{array}{c|c|c} 6I_3 & \begin{matrix}6&6&0\\6&0&6\\0&6&6\end{matrix} & \begin{matrix}6\\6\\6\end{matrix} \\ \hline 0 & 2I_3 & \begin{matrix}2\\2\\2\end{matrix} \\ \hline 0\ \ 0\ \ 0 & 0\ \ 0\ \ 0 & 1 \end{array}\right), \qquad \begin{matrix}\chi_\Sigma(M_{*333}) = 0,\\ \chi(M_{*333}) = 0.\end{matrix}$$

A Maple Programs

These procedures have been confirmed to run on MAPLE 2021.2. They require the use of the `LinearAlgebra` and `MTM` packages for some of their built in functions and can be ran within the same worksheet and on the same matrices without conflict (in fact, it may prove useful to check both the Euler Characteristic and Series Euler Characteristic separately, as they do not always agree; see subsection 2.3.4). One should take care in introducing additional packages; certain functions are often given generic names that are reused by packages and this may break functionality (for example, the `add` function is often reused by name in other algebra packages).

The comparison of χ and χ_Σ could be easily automated by combining these programs into another procedure that prints their differences, if any; or whether they totally agree. The code for the 'regular' Euler characteristic (Appendix A.1 and A.2) was produced (with minor fixes presented here) for my previous dissertation [Edw22] whilst the code for the series Euler characteristic (Appendix A.3) is new. As the primary focus of this paper is the new material, the previous code is presented mostly as-is, with comments and examples left for the reader to explore themselves or find as necessary in my previous dissertation. The new code receives fuller treatment.

A.1 Computing the (co)weightings of finite categories

```
Weightings := proc(ExampleMatrix::Matrix, {coweight::boolean := false})
global free;
local ProcMatrix, RDim, OnesMatrix, RCT;
    if coweight then
        ProcMatrix := transpose(ExampleMatrix);
    else
        ProcMatrix := ExampleMatrix;
    end if;
    RDim := RowDimension(ProcMatrix);
    if RDim <> ColumnDimension(ProcMatrix) then
        error "You did not input an n x n matrix as expected.";
    end if;
    OnesMatrix := Matrix(RDim,1,1);
    RCT := Rank(ProcMatrix) < Rank(<ProcMatrix|OnesMatrix>);
    if coweight and RCT then
        error "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no coweighting.";
    elif RCT then
        error "The rank of the augmented matrix is greater than the coefficient's rank, hence there is no weighting.";
    end if;
    return LinearSolve(ProcMatrix, OnesMatrix, free='x');
end proc:
```

Listing 1: Weightings Procedure

A.2 Computing the Euler Characteristic of a finite category

```
1  EulerChar := proc(Examplematrix::Matrix);
2  global coweight, chi;
3  local W, CW;
4      try
5          (W := simplify(sum(Weightings(ExampleMatrix)))
6      catch
7          "The rank of the augmented matrix is greater than the coefficient's
               rank, hence there is no weighting.":
8      end try;
9      try
10         (CW := simplify(sum(Weightings(ExampleMatrix, coweight))))
11     catch
12         "The rank of the augmented matrix is greater than the coefficient's
               rank, hence there is no coweighting.":
13     end try;
14     if (W=CW) then
15         print("The weightings agree");
16             print(chi = W);
17     else
18     print("The weightings disagree");
19     print(W, CW);
20     end if;
21 end proc:
```

Listing 2: Euler Characteristic Procedure

A.2.1 Code Commentary

When testing the matrix, $\left(\begin{smallmatrix}1&0\\n&m\end{smallmatrix}\right)$, a small error was found with the output. The procedure would claim the weightings disagreed, as they were $(1-\frac{n-1}{m})$ and $(\frac{m-n}{m}+\frac{1}{m})$; weighting and coweighting respectively. These are, of course, the same for any possible input $(m>0)$. The inclusion of `simplify` (lines 5 and 10) has mended this erroneous disagreement by turning both terms into a single fraction where Maple admits they agree, as is expected.

A.3 Computing the Series Euler Characteristic of a finite category

```
SeriesEuler:= proc(inputMatrix::Matrix)
global chi, Sigma, t;
local id, innerMat, detMat, sumAdj, polyEuler;
    id := IdentityMatrix(Dimension(inputMatrix));
    innerMat := id - (inputMatrix - id) * t;
    detMat := Determinant(innerMat);
    sumAdj := add(Adjoint(innerMat));
    return(chi[Sigma] = simplify(sumAdj / detMat))
end proc:
```

Listing 3: Series Euler Characteristic Procedure

A.3.1 Code Commentary

The code is rather straight forward. Here we allow the user to produce a matrix for input, aptly named the 'inputMatrix' which is fed into the SeriesEuler procedure. This matrix comes from the finite category, with elements corresponding to the number of morphisms between each object. Then we simply compute its corresponding formal power series as outlined by the formula 17. This enables us to find the Series Euler characteristic later by running the following command: `eval(SeriesEuler(inputMatrix), t=-1)` on a given matrix with name `inputMatrix`. One might wish to also wrap this with a `simplify` too, in order to reduce repeated terms. Another strong recommendation is the use of `eval(singular(SeriesEuler(inputMatrix)), t=-1)`. We find by inclusion of the `singular` function, we obtain a set of values where the Series Euler characteristic breaks down for matrices containing generic elements. For example, testing on the matrix $\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & n \\ 0 & n & n \end{pmatrix}$ the above command returns: $\{-1 = -1, n = 1\}$ and $\{-1 = -1, n = 0\}$, indicating that both $n = 0$ and $n = 1$ create matrices for which there is either: no series Euler characteristic, or one which *may not* agree with the general case (here, $n = 0$ gives us no χ_Σ nor χ, and $n = 1$ gives us no χ but $\chi_\Sigma = 3/2$; which exists but is certainly not of the form $1/n$ for possible values of n). If one is interested in these examples, they should be checked by a manual substitution or by simply defining `n := x` for a value of interest $x \in \mathbb{N}$ and variable n. The redundant $-1 = -1$ claim is a byproduct of taking $t = -1$ and then substituting in $t = -1$. It is possible for certain matrices to generate nonsense outputs that look something like: $\{-1 = \frac{1}{2}\}$, which suggests that there exists a singularity of n provided $t \neq -1$ but $t = 1/2$ instead; as such, we can safely ignore any such $n = N$ prompt that come alongside this type of false output.

The `simplify` function is used when returning the polynomial fraction as we wish to cancel out any shared terms in the numerator and denominator so as to not cause trouble when evaluating certain polynomials (likely those with $(t + 1)$ roots).

As we discussed in the beginning of this appendix, one should expect to run the traditional Euler characteristic program also, so I have not included any conditions for catching mistaken inputs with the `error` command as before (Appendix A.1), as one is likely to run into issues with that procedure first. In addition, any such errors are unlikely to be useful anyway. It seems that a sensible operator might only encounter an error by first introducing a faulty input, perhaps by incorrectly sizing a matrix or by attempting to introduce a matrix with zero determinant, which by a condition of Lemma 2.43 is not a correct formulation of any such input matrix in the first place.

This procedure is much simpler than the code for the 'regular' Euler characteristic, though their computational complexity has not been considered. Thus far, any matrix tested has ran near instantaneously on relatively modest hardware, so this has not yet been a point of concern. If a notable example of a finite category containing a large number of objects exists then the author would be keen to test the program on it themselves!

B Statement of Originality

This dissertation was written by me, in my own words, except for quotations from published and unpublished sources which are clearly indicated and acknowledged as such. I am conscious that the incorporation of material from other works or a paraphrase of such material without acknowledgement will be treated as plagiarism, according to the University Academic Integrity Policy. The source of any picture, map or other illustration is also indicated, as is the source, published or unpublished, of any material not resulting from my own research.

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