Wallpaper Groups from Scratch

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May 22, 2021

Outline

The Euclidean Plane

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So any mapping f 'preserves distance' in the plane.

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3. Rotations
4. Gliding Reflections

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Those of the form such that for $x, y, a, b \in \mathbb{R}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

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Where M is an orthogonal matrix $(M \cdot M^T = I_2)$

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For brevity, this is the shortest example and just the mappings will be shown

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M_{tr} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$
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Where our $M_{tr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ evidently, this works and to nobody's surprise also an element within the orthogonal **group**.

Reflections

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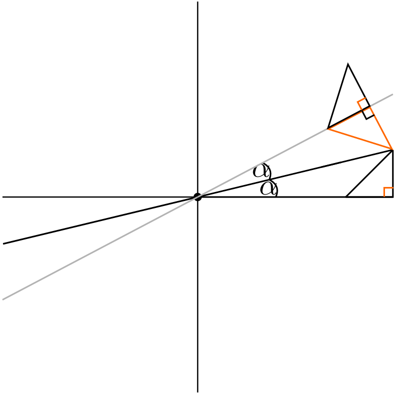

Figure: Visualising the reflection through an angle α of our black triangle. The grey line additionally represents the reflected x-axis.

Rotations

Similarly, a diagram for rotations is equally acceptable.

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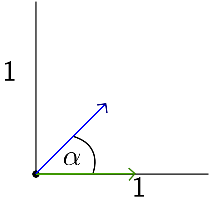

Figure: Illustrating the rotation of a unit vector (in green) by an angle α into the blue vector.

Clearly once again, nothing about distances has seemingly been harmed.

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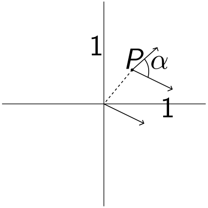

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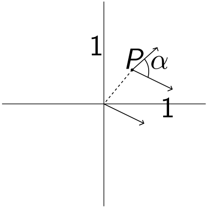

Figure: Illustrating the rotation of a unit vector (in green) by an angle α into the blue vector.

This option has us first translate our vector through the dotted line and then rotate it α degrees about the point P.

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That is all. But where is the identity isometry? You cry out.
It was already there! We could have taken a very boring translation where $a = b = 0$ or even a rotation by 0 degrees. These isometries in fact make up every possible isometry of the Euclidean plane.

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Figure: Oh no.

What is a wallpaper group? No, seriously.

We can define the wallpaper groups to encompass the symmetry-respecting 2-dimensional repeating patterns of the Euclidean plane, given some leeway.

What is a wallpaper group? Formally, please.

We define a discrete subgroup of isometries of the Euclidean plane with two linearly independent translations and we consider the chosen wallpaper group to be all such groups which are the same upto affine transformations of the plane.

On being Discrete

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As though it were planned, we've just discussed these isometries. But we must go back to pick up on the word 'discrete' and what we meant by it here. We are concerned with it in the topological sense. That's to say that for each point P in the plane, there exists a finite neighbourhood of P such that some nearby points Q in the orbit of P happen to intersect at finitely many points.

Why those properties?

As eluded to at the start of this section. The naming is no mistake. Suppose we had a (slightly nicer looking) pattern on part of someones wall.

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Figure: Not bad.

Well what if they liked the pattern and wanted it across the rest of their wall? We clearly want some way to extend the pattern.

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Continue this indefinitely, and you see why it's quite nice that we have that there are two linearly independent translations so that we can use to span a plane infinitely.

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Then the discreteness property addresses a problem of scale of patterns. It doesn't really make sense to have an infinitely small pattern in the plane or have a pattern on an infinitely small wall...

So we've then handled both extreme cases.

Outline

The Torus

Jumping into a bit of topology; we have that a torus can be made out of a Euclidean space by taking opposing edges of the plane and aligning them.

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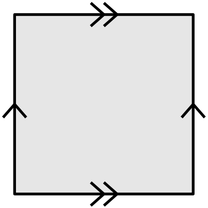

Figure: Match opposing arrows and join them together to form a torus.

Euler Characteristics

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Then we make use of the fact that we can define for closed-orientable surfaces, their Euler characteristic are of the form

$$\chi = 2 - 2g$$

where g denotes the 'genus' or - more intuitively - the number of punctures or holes in the surface.

Euler Characteristics

Clearly then. A torus has Euler characteristic

$$\chi = 2 - 2(1) = 0.$$

Other Properties of Tori

We will next introduce two propositions without proof that allow us to take a quotient mapping of G/Λ acting on the torus T_Λ.

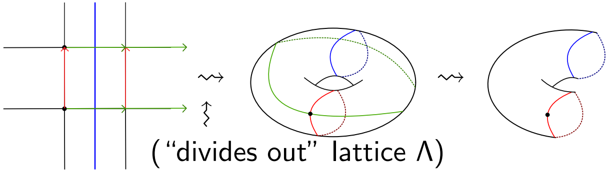

Figure: Visualisation of this process.

Propositions

Proposition 1

The translations in G form a lattice Λ such that

$$\Lambda = \mathbb{Z} \cdot \vec{t_1} + \mathbb{Z} \cdot \vec{t_2}$$

Where $\vec{t_1}$ and $\vec{t_2}$ denote two linearly independent translations.

Propositions

Proposition 2

We are able to divide out the lattice Λ of translations of G from the torus T_Λ and the action of G on $\mathbb{E}^2$ 'descends' to an action of G/Λ on T_Λ.

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Corollary

This also implies that Λ is a normal subgroup of G. Namely;

$$\Lambda \trianglelefteq G.$$

Between these two propositions we are now able to take our wallpaper groups and map them onto tori.

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In this application, we are using them to allow us to ‘glue’ in discs to edges (boundaries) and to keep properties that we want.

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On compact topological surfaces with boundaries, there always exists a finite CW Complex.

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So then these have finite CW-Complexes. A fact we will return to later.

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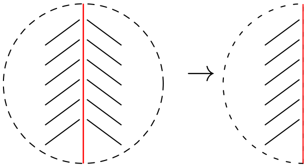

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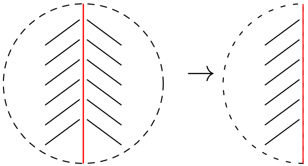

Figure: Visualisation of the 'folding' of points by a reflection in the plane.

We can unfold it back to the original but in the meantime we have captured the information we need from it in a more dense manner.

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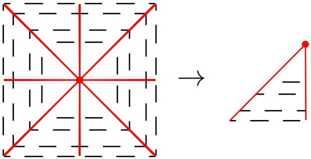

Figure: Folding about a corner point.

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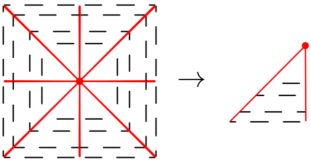


Figure: Folding about a corner point.

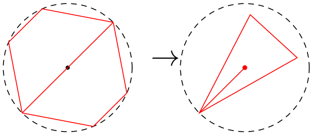


Figure: Folding about a cone point.

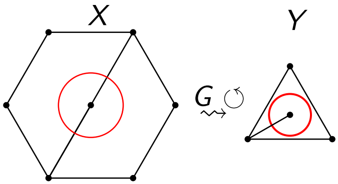
Outline

Fractional Euler Characteristics

We make a return to Euler characteristics but now we are interested in what happens to it as a result of the impact of the CW-Complexes.

Cone points

If we have the following arrangement for a cone point of order 2:



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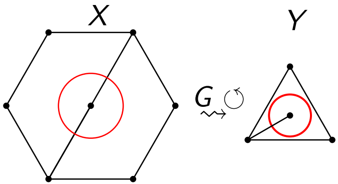

Figure: Topological spaces X and Y for the case of a Corner point of order 2.

We can track its original Euler characteristic

$$\chi(X) = V - E + F = 7 - 8 + 2 = 1$$

and the the Euler Characteristic of Y,

$$\chi(Y) = 4 - 4 + 1 = 1.$$

'Defects'

Then tracking the additional contributions brought about by counting vertices and edges too many times. We have about the central cone point:

$$\chi(X) = 1 - \left(\frac{1}{2} + \frac{1}{2}\right) + (a + a)$$

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and looking at Y,

$$\chi(Y) = 1 - \frac{1}{2} + a$$

But now we can see there's a difference between $\frac{\chi(X)}{2}$ and $\chi(Y)$! This is what we call the defect (in this case, $\frac{\chi(X)}{2} = \chi(Y) - \frac{1}{2}$)

Corner Point case

Taking a quick look towards an example with corner points involved.

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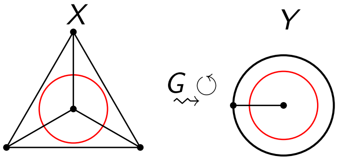

Figure: Topological spaces X and Y for the case of a Corner point of order 3.

Defects

Much the same, skipping to our ‘defected characteristics’.

$$\chi(X) = 1 - 6 \times \frac{1}{2} + 6 \times a, \qquad \chi(Y) = 1 - 2 \times \frac{1}{2} + a$$

Then

$$\frac{\chi(X)}{6} = \frac{1}{6} - \frac{1}{2} + a = \chi(Y) - \left(1 - \frac{1}{6}\right) + \frac{1}{2} = \chi(Y) - \frac{5}{12}$$

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Which happens to look slightly different from before, but now is of the form:

$$\text{Defect for } n\text{-many corner points} = \frac{1}{2} - \frac{1}{2n}$$

Calculating it

Now because we are acting on tori and orbifolds. We need not be concerned with any difference of faces, so we have now found the two ways in which we can alter the Euler characteristic and they can be generalised as follows:

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Definition

Let X be a compact surface, $G \circlearrowleft X$ the group acting on X and Y the orbifold of this action. Then:

$$\frac{\chi(X)}{|G|} = \chi(Y) - \sum_{p.\ \substack{\text{Cone points} \\ \text{of order } n}} \left(1 - \frac{1}{n}\right) - \sum_{p.\ \substack{\text{Corner points} \\ \text{of order } n}} \left(\frac{1}{2} - \frac{1}{2n}\right)$$

Outline

Classification of Compact Surfaces

We now have a mostly complete picture.

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But now we take use of the following theorem:

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Theorem

"Every orientable compact surface is homeomorphic either to a sphere or to a connected sum of tori. Every nonorientable compact surface is homeomorphic either to a projective plane, or a Klein bottle, or the connected sum of a projective plane or a Klein bottle with some tori." [Gallier and Xu, 2013]

Types of Orbifolds

With this theorem we can now see that we have a few options for our potential orbifolds, Y.

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These are:

- Disks,
- Annulus,
- Spheres,
- Möbius strips,
- Klein Bottles and
- The Real Projective plane.

How we use this theorem

By combining the defect formula with this theorem then we can deduce a bound of sorts for our Euler characteristics and figure out which surfaces we can map wallpapers to

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By combining the defect formula with this theorem then we can deduce a bound of sorts for our Euler characteristics and figure out which surfaces we can map wallpapers to, and how!

$$\chi(\text{Disk}) = 1$$
$$\chi(\text{Annulus}) = 0$$
$$\chi(\text{Sphere}) = 2$$
$$\chi(\text{Torus}) = 2$$

$$\chi(\text{Möbius strips}) = 0$$
$$\chi(\text{Klein Bottles}) = 0$$
$$\chi(\text{Real Projective Plane}) = 1$$

Brief aside to Orbifold Characters

We will simply list these explicitly with short descriptions:

1. Numbers added after an $*$: Indicative of Corner points.
2. Numbers preceding an $*$ (if there is one) : Indicative of Cone points.
3. $\times$: Depicts the inclusion of a crosscap (making the orbifold nonorientable).
4. $\circ$: Depicts the inclusion of a handle.

Candidates with $\chi = 0$

We can then consider that any of the surfaces with $\chi = 0$ work without the use of neither corner nor cone points and these are given as follows in orbifold notation:

Annulus,	Möbius strips,	Klein Bottles.
$**$	$*\times$	$\times\times$

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$$\underset{**}{\text{Annulus,}} \quad \underset{*\times}{\text{Möbius strips,}} \quad \underset{\times\times}{\text{Klein Bottles.}}$$

(This is also the case for the Torus, which has no cone or corner points though has $\chi = 2$ and is represented of course by just a handle $\circ$.)

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As follows:

$$
\begin{array}{ccccccc}
\text{Disk} & \rightsquigarrow & *632, & *442, & *333, & *2222 \\
 & & & \downarrow & \downarrow & \downarrow \\
 & & & 4*2 & 3*3 & 2*22 \\
 & & & & & \downarrow \\
 & & & & & 22*
\end{array}
$$

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Then considering some other examples with non-zero Euler Characteristic, there are a few ways to reduce theirs with corner and cone points.
As follows:

$$\begin{array}{ccccc} \text{Disk} \rightsquigarrow & *632, & *442, & *333, & *2222 \\ & & \downarrow & \downarrow & \downarrow \\ & & 4*2 & 3*3 & 2*22 \\ & & & & \downarrow \\ & & & & 22* \end{array}$$

In particular, we have for say, $*632$ corner points of order 6,3 and 2 respectively, hence

$$\chi(*632) = \frac{5}{12} + \frac{1}{3} + \frac{1}{4} = 1$$

Candidates for the Sphere and Real Projective Plane

Similarly then for the sphere which has no corner points because we have no boundary!

$$\text{Sphere} \rightsquigarrow 632,\ 432,\ 333,\ 2222$$

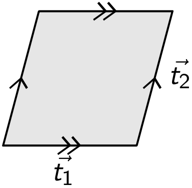
and finally:

$$\text{Real Projective Plane} \rightsquigarrow 22\times$$

Between these then, we have found what we will determine to be all 17 orbifolds!

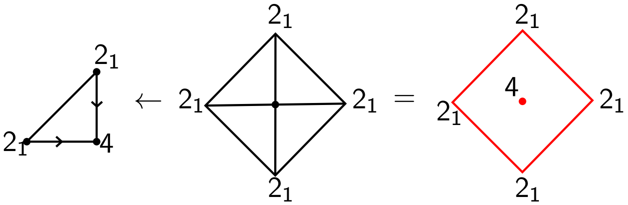
The Torus

Let's start with the torus:



We have in this case that all translations generated by $\vec{t_1}, \vec{t_2}$ will trivially produce wallpapers spanning the plane.

A simple example on the disk with orbifold notation $4 * 2$.



We have around the central corner point of order 4, cone points of order 2 in the cardinal directions which we can use to generate the rest of the plane.

Conclusion

We can continue to show that these exist for the remaining orbifolds and exhaustively deduce that all seventeen proposed candidates are indeed suitable wallpaper groups.

The End!